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INDUCED REPRESENTATIONS OF C^*-ALGEBRAS AND COMPLETE POSITIVITY[1]

BY

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ABSTRACT. It is shown that *representations may be induced from one C^*-algebra B to another C^*-algebra A via a vector space equipped with a completely positive B-valued inner product and a *representation of A. Theorems are proved on induction in stages, on continuity of the inducing process and on completely positive linear maps of finite dimensional C^*-algebras and of group algebras.

Introduction. We are concerned with applying to C^*-algebras the following notion of induced representations of rings. Let A and B be two rings, X a left A-module and right B-module. Given a representation of B, i.e., a left B-module $\mathfrak{K}$, the representation induced by X and $\mathfrak{K}$ is the inherited action of A on $X \otimes_B \mathfrak{K}$. This is a construction used by Higman [9] for the case in which $X = A$ and B is a subring of A and in this form is readily seen to include the usual induced representations of finite groups. In [13] Rieffel showed that this construction makes sense for Banach space representations of Banach algebras. When A and B are C^*-algebras the problem is to insure that when one begins with a *representation of B on a Hilbert space $\mathfrak{K}$ one can equip $X \otimes_B \mathfrak{K}$ with an inner product so that the induced representation of A is a *representation on a Hilbert space. Rieffel [14] and Fell [8] showed how this could be done if X came equipped with a B-valued inner product satisfying certain requirements. Such spaces are similar to those called B-*modules in §IV below. Among their properties is the fact that their inner products must always be completely positive, in the sense defined in §III. In the following we drop the assumption that X be a *module and assume in its place merely that its inner product is completely positive.

Experience with C^*-algebras suggests that an algebraic property can often be replaced by an appropriate order property, such as complete positivity, if the attendant shifts in perspective and technique are allowed. This principle holds in the case of induced representations, as §§I–III will show. In §I we establish some basic properties of vector spaces equipped with C^*-algebra

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valued inner products, called B-spaces if B is the C^*-algebra. In particular we show that when a B-space is equipped with a certain natural norm the inner product is jointly continuous. This was also shown by Rieffel [**14**] and Paschke [**12**] but the proofs given by them required that the B-space be, essentially, a B-*module. In §II we define *representations of pre-C^*-algebras on B-spaces and show how to construct such a *representation of a pre-C^*-algebra A given a relatively bounded positive linear map of A into B. The construction is analogous to the construction of Hilbert space *representations from relatively bounded positive linear functionals. Unlike Hilbert space *representations of a C^*-algebra, however, B-space *representations need not be direct sums of *representations constructed from bounded positive linear maps. §III shows how to induce *representations from B to A given a *representation of A on a completely positive B-space. We also show that although we have made definitions and constructions in terms of pre-C^*-algebras as in [**14**] rather than in terms of C^*-algebras, the additional generality afforded by this is, as far as the inducing process is concerned, illusory.

The notion of induced representation arrived at in §III has gotten rather far away from the well-known construction for groups. However, by providing generalizations of some important properties of induced representations of groups, parts of §§IV–VI may also provide some justification for borrowing the name. In §IV we give an induction-in-stages theorem similar to the one in [**14**] but in which complete positivity again serves in place of the *module assumption. This is used to point out how one can obtain a B-*module which will induce the same representations as a given B-space. We also define a property, called the Blattner property, which is implied by the *module assumption and in the finite dimensional case is almost equivalent to it. In §V we prove that the inducing process preserves weak containment, generalizing the theorem of Fell [**7**] for induced representations of groups. This is used to give a characterization of the support of the collection of all *representations induced by a given B-space. §VI gives a complete classification, up to inductive equivalence, of the completely positive linear maps of one finite dimensional C^*-algebra into another. These results are used to prove a reciprocity theorem which includes the Frobenius reciprocity theorem for finite groups.

In §§VII and VIII we turn to locally compact groups. The first part of §VII is a quick review of the construction of the usual Mackey-Blattner induced representations based on the ideas employed by Fell [**8**] and Rieffel [**14**]. In the second part we look at the inducing of representations from a compact subgroup in a different way. §VII gives an integral representation of bounded positive linear maps from a group algebra into a C^*-algebra which establishes an interesting difference between positive and completely positive maps.

Terminology. For the most part we follow the standard terminology for C^*-algebras as in Dixmier [6] or Sakai [16]. The following exceptions will be convenient in various places.

Following Rieffel [14], dense *subalgebras of C^*-algebras will be called pre-C^*-algebras. Of course every normed algebra with involution which satisfies all the axioms of a C^*-algebra except completeness is a pre-C^*-algebra since its completion is a C^*-algebra. An element of a pre-C^*-algebra will be called positive if it is a positive element of the completion.

If A is a C^*-algebra, a pair $(\pi, \mathfrak{K})$ will be called a *representation of A if $\mathfrak{K}$ is a pre-Hilbert space and π extends to a *representation in the usual sense on the Hilbert space completion $\tilde{\mathfrak{K}}$. A *representation of a pre-C^*-algebra will mean the restriction of a *representation in the above sense of its completion. Two such *representations $(\pi_1, \mathfrak{K}_1)$ and $(\pi_2, \mathfrak{K}_2)$ will be called unitarily equivalent if their extensions $(\pi_1, \tilde{\mathfrak{K}}_1)$ and $(\pi_2, \tilde{\mathfrak{K}}_2)$ are unitarily equivalent.

I. C^*-algebra valued inner products.

Definition. Let B be a pre-C^*-algebra. A complex vector space X will be called a B-space if it is equipped with a B-valued inner product, i.e., a map $\langle\langle\cdot, \cdot\rangle\rangle$ from $X \times X$ into B such that

(1) $\langle\langle\cdot, \cdot\rangle\rangle$ is linear in the first entry and conjugate linear in the second,

(2) $\langle\langle x, x\rangle\rangle \geqslant 0$, i.e., $\langle\langle x, x\rangle\rangle$ is always a positive element of B, for any x in X.

X and $\langle\langle\cdot, \cdot\rangle\rangle$ will be called definite if $\langle\langle x, x\rangle\rangle = 0$ only when $x = 0$ and semidefinite to emphasize that they are not necessarily definite.

Semidefinite B-spaces are called pre-B-Hilbert spaces in [14] and, subject to further conditions, appear as pre-B-Hilbert modules in [12].

If X is a semidefinite B-space and φ is a positive linear functional on B, then $x, y \mapsto \langle x, y\rangle_\varphi = \varphi(\langle\langle x, y\rangle\rangle)$ defines a positive semidefinite inner product on X. Such an inner product must satisfy the Schwarz inequality $|\langle x, y\rangle_\varphi|^2 \leqslant \langle x, x\rangle_\varphi \langle y, y\rangle_\varphi$. Since $\langle\langle x, x\rangle\rangle = 0$ if and only if $\varphi(\langle\langle x, x\rangle\rangle) = 0$ for all positive linear functionals φ on B, the Schwarz inequality shows that the set N of elements x of X such that $\langle\langle x, x\rangle\rangle = 0$ forms a linear subspace. The value of $\langle\langle\cdot, \cdot\rangle\rangle$ on X/N is well defined and X/N forms a definite B-space.

Any B-valued inner product on a vector space X is necessarily Hermitian, i.e., $\langle\langle x, y\rangle\rangle = \langle\langle y, x\rangle\rangle^*$ for all x and y in X. To prove this, let φ be any continuous positive linear functional on B, note that $\langle x, y\rangle_\varphi = \overline{\langle y, x\rangle_\varphi}$, and conclude from the fact that φ is Hermitian that $\varphi(\langle\langle x, y\rangle\rangle) = \varphi(\langle\langle y, x\rangle\rangle^*)$. Since this is true for all such φ, we have $\langle\langle x, y\rangle\rangle = \langle\langle y, x\rangle\rangle^*$.

The positive linear functionals on B can be used to define a convenient topology on X as follows. Any positive linear functional φ defines a seminorm on X by $\|x\|_\varphi = (\langle x, x\rangle_\varphi)^{1/2}$. Let $\|x\|_*$ be the seminorm given by taking the supremum of the seminorms $\|\cdot\|_\varphi$ over all positive linear functionals φ of norm 1. This seminorm will of course equal $\|\langle\langle x, x\rangle\rangle\|^{1/2}$, where $\|\cdot\|$ denotes the C^*-algebra norm on B. Notice that $\|x\|_* = 0$ if and only if $\langle\langle x, x\rangle\rangle = 0$ so that $\|\cdot\|_*$ will be a norm on X precisely when $\langle\langle \cdot, \cdot\rangle\rangle$ is definite.

A B-space will always be assumed to be equipped with the topology determined by $\|\cdot\|_*$.

PROPOSITION. *If* X, $\langle\langle \cdot, \cdot\rangle\rangle$ *is any* B*-space, then* $\|\langle\langle x,y\rangle\rangle\| \leqslant 4\|x\|_*\|y\|_*$, *for all* x *and* y *in* X.

PROOF. If φ is a state on B, then for any x and y in X, $\varphi(\langle\langle x,y\rangle\rangle) \leqslant \|x\|_\varphi\|y\|_\varphi$ by the Schwarz inequality. If τ is any continuous linear functional on B of norm 1, τ can be written in the form $\tau = \alpha_1\varphi_1 - \alpha_2\varphi_2 + i(\alpha_3\varphi_3 - \alpha_4\varphi_4)$ where each φ_i is a state and each α_i is a nonnegative number less than or equal to 1. Therefore

$$\tau|(\langle\langle x,y\rangle\rangle)| \leqslant \sum_{i=1}^{4} |\varphi_i(\langle\langle x,y\rangle\rangle)| \leqslant \sum_{i=1}^{4} \|x\|_{\varphi_i}\|y\|_{\varphi_i}$$
$$\leqslant 4\|x\|_*\|y\|_*.$$

Consequently $\|\langle\langle x,y\rangle\rangle\| \leqslant 4\|x\|_*\|y\|_*$. □

The proposition shows that $\langle\langle \cdot, \cdot\rangle\rangle$ is necessarily continuous. It may therefore always be extended continuously to the Banach space completion $\tilde{X}$ of X as long as the extension is allowed to take its values in the completion of B. The extension will be positive since the maps $x \mapsto \varphi(\langle\langle x, x\rangle\rangle)$ are continuous for all states φ.

Linear subspaces of B-spaces are again B-spaces under the inherited B-valued inner product. Subspaces of B-spaces need not be orthocomplemented, i.e., if Y is a subspace of X and $Y^\perp = \{x \in X | \langle\langle x,y\rangle\rangle = 0$ for all y in $X\}$ it need not be the case that $Y + Y^\perp$ is dense in X. For a simple example, take X and B both to be the C^*-algebra of all continuous functions on the closed unit interval [0, 1] and define $\langle\langle f, g\rangle\rangle = \bar{g}f$. Let Y be the subspace consisting of all those functions which vanish at 0. Then Y is closed and $Y^\perp = \{0\}$.

Despite the formal analogies between B-spaces and Hilbert spaces, properties such as the above together with the decomposability of B itself make their decomposition theory somewhat more complicated than that of Hilbert spaces.

Suppose that B contains two projections e_1 and e_2 which are mutually orthogonal and central and suppose that $e_1 + e_2$ is the identity on the range of the B-valued inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on X. Define new B-valued inner products by

$$\langle\langle x, y \rangle\rangle_i = e_i \langle\langle x, y \rangle\rangle, \qquad i = 1, 2.$$

Then $\langle\langle x, y \rangle\rangle = \langle\langle x, y \rangle\rangle_1 + \langle\langle x, y \rangle\rangle_2$ and $\langle\langle \cdot, \cdot \rangle\rangle_1$ and $\langle\langle \cdot, \cdot \rangle\rangle_2$ are orthogonal in the sense that

$$\langle\langle x_1, y_1 \rangle\rangle_1 \langle\langle x_2, y_2 \rangle\rangle_2 = \langle\langle x_1, y_1 \rangle\rangle \langle\langle x_2, y_2 \rangle\rangle e_1 e_2 = 0.$$

So $\langle\langle \cdot, \cdot \rangle\rangle$ is an orthogonal sum of two presumably simpler B-valued inner products both with the same domain X.

Before considering decompositions of X we define what will be meant by the direct sum of a collection of B-spaces.

DEFINITION. Let B be a pre-C^*-algebra, X_i, $\langle\langle \cdot, \cdot \rangle\rangle_i$ a collection of B-spaces indexed by some set I. By their direct sum we mean the vector space direct sum, consisting of all formal sums Σx_i of families of vectors x_i which are nonzero for only a finite number of i's, together with the B-valued inner product $\langle\langle \Sigma x_i, \Sigma x_j \rangle\rangle = \Sigma \langle\langle x_i, y_i \rangle\rangle$.

In order that a B-space X, $\langle\langle \cdot, \cdot \rangle\rangle$ be the B-space direct sum of a collection of subspaces X_i equipped with their inherited inner products it is necessary and sufficient that X be the vector space direct sum of the subspaces and that they be mutually orthogonal, $\langle\langle X_i, X_j \rangle\rangle = \{0\}$ if $i \neq j$.

Suppose that a B-space X, $\langle\langle \cdot, \cdot \rangle\rangle$ is the direct sum of its subspaces X_1 and X_2. Since $X_1 \cap X_2 = \{0\}$ and $X_1 + X_2 = X$, we can define projections E_1 and E_2 by $E_i(x) = x_i$, $i = 1, 2$, where $x = x_1 + x_2$ is the unique expression for x with x_1 in X_1 and x_2 in X_2. Each E_i is bounded and idempotent. Since X_1 and X_2 are orthogonal, we can also conclude that $\langle\langle E_i x, E_i x \rangle\rangle \leq \langle\langle x, x \rangle\rangle$ for any x in X and that $\langle\langle E_i x, y \rangle\rangle = \langle\langle x, E_i y \rangle\rangle$ for all x and y in X. These properties insure that each E_i is in acceptable operator on X in the sense defined below.

DEFINITION. Let X, $\langle\langle \cdot, \cdot \rangle\rangle$ be a B-space. An acceptable operator on X is a linear map T of X into itself such that

(1) There is a nonnegative constant k such that $\langle\langle Tx, Tx \rangle\rangle \leq k^2 \langle\langle x, x \rangle\rangle$ for all x in X. The smallest such k is denoted $|T|$.

(2) There is another linear map T^* from X into itself which satisfies (1) and for which $\langle\langle Tx, y \rangle\rangle = \langle\langle x, T^* y \rangle\rangle$ for all x and y in X. T^* is called an adjoint of T.

This definition is taken from **[14]** where such operators are called bounded operators on X. The name is changed here because although an acceptable operator is bounded as an operator on the seminormed space X (its bound is certainly $\leq |T|$), an operator on X which is bounded need not be acceptable.

Paschke gave an example in [12] of an operator on a complete B-space which is bounded and satisfies condition (1) but not condition (2).

Denote the set of acceptable operators on X by $\mathcal{Q}(X)$. In [14] Rieffel showed that $\mathcal{Q}(X)$ is a seminormed algebra with $|\cdot|$ as seminorm and composition as multiplication and that if $\langle\langle\cdot, \cdot\rangle\rangle$ is definite adjoints are unique and $\mathcal{Q}(X)$ is a pre-C^*-algebra.

$\mathcal{Q}(X)$ is intended to be an analogue of the algebra of bounded operators on a Hilbert space. As the remarks preceding the definition indicate, it shares with such an algebra the property that if X is the direct sum of two orthogonal subspaces, then there exist projections E_1 and E_2 onto those subspaces which are projections in $\mathcal{Q}(X)$ and which satisfy $E_1E_2 = 0 = E_2E_1$ and $E_1 + E_2 = I$ (the identity map I on X is clearly in $\mathcal{Q}(X)$). Conversely, if E_1 is a projection in $\mathcal{Q}(X)$ and $E_2 = I - E_1$, then X is easily seen to be the B-space direct sum of its orthogonal subspaces E_1X and E_2X.

II. *Representations on B-spaces and positive linear maps. Let A and B be pre-C^*-algebras and X, $\langle\langle\cdot, \cdot\rangle\rangle$ a B-space.

DEFINITION. A *representation of A on the B-space X is a left-module action of A on X satisfying

(1) $\langle\langle ax, y\rangle\rangle = \langle\langle x, a^*y\rangle\rangle$ for all x and y in X,

(2) $\langle\langle ax, ax\rangle\rangle \leq \|a\|^2\langle\langle x, x\rangle\rangle$ for all x in X and a in A.

One immediate consequence of the definition is that each element of A acts on X as an acceptable operator, so that a *representation of A on X is simply a *homomorphism of A into the algebra of acceptable operators on X.

If X is not definite, condition (2) shows that the subspace of vectors of norm zero form a submodule, so the action of A on the definite quotient B-space is again a *representation.

Condition (2) also indicates that the map $a, x \mapsto ax$ from $A \times X$ into X is continuous ($\|ax\|_* \leq \|a\|\,\|x\|_*$). If X is definite this map may therefore be extended in a unique continuous manner to the completions $\tilde{A} \times \tilde{X} \to \tilde{X}$. One can easily check that this map defines a *representation of $\tilde{A}$ on $\tilde{X}$. Since the action on X of $\mathcal{Q}(X)$, the pre-C^*-algebra of acceptable operators, determines a *representation of $\mathcal{Q}(X)$, we can define an isometric *homomorphism of the completion $\tilde{\mathcal{Q}}(X)$ into $\mathcal{Q}(\tilde{X})$. If X is complete as well as definite we may therefore conclude that $\mathcal{Q}(X)$ is complete, i.e., a C^*-algebra.

It is interesting to note that, in analogy with the case of *representations on Hilbert spaces, if A is a C^*-algebra then condition (2) is superfluous. To see this first notice that we can assume that A has an identity, since if it does not we can adjoin one and extend the *representation on X to the larger algebra. Now if a is any element of A, $\|a\|^2 - a^*a$ is positive, hence has a positive square root h in A. Then for any x in X, $\langle\langle hx, hx\rangle\rangle = \langle\langle h^2x, x\rangle\rangle$ is a positive element of B and equals $\|a\|^2\langle\langle x, x\rangle\rangle - \langle\langle ax, ax\rangle\rangle$, which implies condition (2).

If A is a pre-C^*-algebra with a *representation on a B-space X and x is a fixed element of X, one can define a linear map Φ from A into B by $\Phi(a) = \langle\langle ax, x\rangle\rangle$. Φ is necessarily bounded since

$$\|\langle\langle ax, x\rangle\rangle\| \leqslant 4\|ax\|_*\|x\|_* \leqslant 4\|a\|\;\|x\|_*^2 .$$

Furthermore Φ takes elements of the form a^*a to positive elements of B and so satisfies the following definition.

DEFINITION. If A and B are pre-C^*-algebras, a linear map Φ from A into B will be called positive if $\Phi(a^*a)$ is a positive element of B for any a in A.

Any such positive linear map determines a B-valued inner product on A by $\langle\langle a_1, a_2\rangle\rangle = \Phi(a_2^*a_1)$. In analogy with the procedure for positive linear functionals one would like to define a *representation of A on the B-space A, $\langle\langle\cdot, \cdot\rangle\rangle$. Left multiplication defines an action of A on A which obviously satisfies condition (1) for a *representation. In order to conclude that condition (2) will be satisfied we must know that for any a and a_0 in A

$$\Phi(a_0^*a^*aa_0) \leqslant \|a\|^2\Phi(a_0^*a_0).$$

This is the case for maps satisfying the following definition.

DEFINITION. A positive linear map Φ from one pre-C^*-algebra A into another is relatively bounded if for each a in A the map $\Phi_a(a_1) = \Phi(a^*a_1a)$ is bounded on elements of the form $a_1 = b^*b$.

This is essentially the same definition as in **[14]**.

LEMMA. *If Φ is a relatively bounded positive linear map from a pre-C^*-algebra A into a pre-C^*-algebra B, then for any a and b in A*

$$\Phi(a^*b^*ba) \leqslant \|b\|^2\Phi(a^*a).$$

PROOF. Let φ be a state of B and a an element of A. Then $\varphi \circ \Phi_a$ is a positive linear functional on A. If A does not have an identity we may adjoin one and extend $\varphi \circ \Phi_a$ by setting $\varphi \circ \Phi_a(1) = \varphi \circ \Phi(a^*a)$. The extension will again be positive since for any b in A

$$\varphi \circ \Phi_a((1 + b)^*(1 + b)) = \varphi \circ \Phi((a + ba)^*(a + ba)).$$

From the Schwarz inequality we know that

$$|\varphi \circ \Phi_a(b)|^2 \leqslant \varphi \circ \Phi_a(1)\varphi \circ \Phi_a(b^*b) = \varphi \circ \Phi(a^*a)\varphi \circ \Phi_a(b^*b)$$

(see, e.g., 26H of **[11]**). Since Φ is relatively bounded there is some constant K such that $\|\Phi_a(b^*b)\| \leqslant K\|b\|^2$ so

$$|\varphi \circ \Phi_a(b)|^2 \leqslant (\varphi \circ \Phi(a^*a)K)\|b\|^2,$$

and we see that $\varphi \circ \Phi_a$ is bounded. Its norm is therefore given by its value at

1, $\varphi \circ \Phi(a^*a)$, so

$$\varphi \circ \Phi_a(b^*b) \leqslant \|b\|^2 \varphi \circ \Phi(a^*a).$$

Since it is true for any state φ the lemma is proved. □

Assume now that $\Phi: A \to B$ is a relatively bounded positive linear map, A and B being pre-C^*-algebras. From the lemma we can conclude that the action of A on the B-space A, $\langle\langle\cdot, \cdot\rangle\rangle$ determined by Φ as above is a *representation. If Φ is not bounded we cannot hope to find a B-space and *representation of A in terms of which Φ can be written in the form $\Phi(a) = \langle\langle ax, x\rangle\rangle$. Even if Φ is bounded the B-space A, $\langle\langle\cdot, \cdot\rangle\rangle$ may not be large enough to accommodate such a representation of Φ, even if the B-space is completed. Of course, if A has an identity 1, the difficulty vanishes since one can always take $\Phi(a) = \langle\langle a1, 1\rangle\rangle = \Phi(1a1)$. So in order to conclude that Φ can be written in such a form for some B-space *representation we need only show that Φ can be extended to a relatively bounded positive linear map on a C^*-algebra with identity which contains A. Since a W^*-algebra always has an identity, this is a consequence of the following lemma.

Lemma. *Let Φ be a bounded positive linear map from one pre-C^*-algebra A into another B. Let $W^*(A)$ and $W^*(B)$ denote the enveloping W^*-algebras of their completions. Then Φ can be extended to a bounded positive linear map $\tilde{\Phi}$: $W^*(A) \to W^*(B)$.*

Proof. Identify $W^*(A)$ with the bidual of A and identify A with its canonical image in the bidual. Similarly for B. Let $\tilde{\Phi}$ be the second transpose of Φ. If α is any positive element of $W^*(A)$ we must show that $\tilde{\Phi}(\alpha)$ is a positive element of $W^*(B)$. It will suffice to show that if φ is any normal positive linear functional on $W^*(B)$, then $\varphi(\tilde{\Phi}(\alpha))$ is nonnegative. Since the normal positive linear functionals on $W^*(B)$ may be identified with the bounded positive linear functionals on B, we may as well assume that φ is in B^*. Then $\tilde{\Phi}(\alpha)(\varphi) = \alpha(\Phi'(\varphi))$, where Φ' denotes the transpose of Φ. But $\Phi'(\varphi)$ is a positive linear functional on the completion of A and therefore determines a positive linear functional on $W^*(A)$. Thus $\tilde{\Phi}(\alpha)(\varphi) \geqslant 0$. □

Proposition. *Let A and B be pre-C^*-algebras. Then $\Phi: A \to B$ is a bounded positive linear map if and only if there is a $W^*(B)$-space X, $\langle\langle\cdot, \cdot\rangle\rangle$ and a *representation of A on X such that $\Phi(a) = \langle\langle ax, x\rangle\rangle$ for some x in X.* □

We conclude this section with a simple example. Let A be a C^*-algebra and $\Phi: A \to A = B$ the identity map of A. The A-space determined by Φ is just A equipped with the inner product $\langle\langle a_1, a_2\rangle\rangle = a_2^* a_1$. Clearly $\langle\langle\cdot, \cdot\rangle\rangle$ is definite, $\|a\|_* = \|a\|$ and the A-space A is complete. The algebra $\mathcal{Q}(A)$ of acceptable operators for this case has been studied under other names. In [4], Busby defined a double centralizer of a C^*-algebra A to be a pair of maps

(T, T') of A into itself such that $aT(b) = T'(a)b$ for all a and b in A. In terms of the inner product defined above, this condition may be rewritten $\langle\langle T(b), a\rangle\rangle = \langle\langle b, T'(a^*)^*\rangle\rangle$. If we define a new map by $T^*(a) = T'(a^*)^*$ it is almost apparent that T is an acceptable operator–we need only prove that T and T^* possess the "boundedness" property $\langle\langle Ta, Ta\rangle\rangle \leqslant k^2\langle\langle a, a\rangle\rangle$. In order to prove this it is convenient to make use of the equivalent definition of double centralizers given by Akemann, Pedersen and Tomiyama in **[1]**. Busby showed that the collection $M(A)$ of double centralizers forms a C^*-algebra with identity. In **[1]**, $M(A)$ is called the multiplier algebra and defined to be the largest C^*-subalgebra of $W^*(A)$ in which A is an ideal.

(That the two definitions of $M(A)$ are equivalent may be seen from the following argument. Given a double centralizer (T, T') of A, extend to the pair $(\tilde{T}, \tilde{T}')$ of double transpose maps of $W^*(A)$ into itself. From the weak* continuity of $\tilde{T}$ and $\tilde{T}'$ and of left and right multiplication in $W^*(A)$ (cf. **[16**, 1.7.8]) one can see that the pair $(\tilde{T}, \tilde{T}')$ forms a double centralizer of $W^*(A)$. Now according to **[4]** the double centralizer algebra of a C^*-algebra with identity is isomorphic to the given algebra in such a way that any pair (T, T') corresponds to left and right multiplication by some element of the algebra. Therefore the double centralizer algebra of A may be identified with a certain C^*-subalgebra of $W^*(A)$ in which A is an ideal. From the maximal nature of the double centralizer algebra one concludes that it is the largest such C^*-subalgebra of $W^*(A)$.)

So any double centralizer of A corresponds to a pair of maps (L_α, R_α) consisting of left and right multiplication by some element α of $W^*(A)$. Then we have

$$\begin{aligned}\langle\langle L_\alpha a, L_\alpha a\rangle\rangle &= (L_\alpha a)^* L_\alpha a = a^*\alpha^*\alpha a \\ &\leqslant \|\alpha\|^2 a^*a = \|\alpha\|^2\langle\langle a, a\rangle\rangle\end{aligned}$$

for any a in A. Similarly for $L_\alpha^* = L_{\alpha^*}$. Thus each double centralizer determines an acceptable operator. Conversely, an acceptable operator defines a double centralizer (T, T') where $T'(a) = T^*(a^*)^*$. We may therefore identify the algebra of acceptable operators with $M(A)$.

Suppose now that A is the algebra $C_0(\Omega)$ of continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space Ω. According to **[1]**, the multiplier algebra is $C_b(\Omega)$, the bounded continuous functions on Ω. We may therefore identify the algebra of acceptable operators on the A-space A with $C_b(\Omega)$, acting on $C_0(\Omega)$ by pointwise multiplication. If we assume further that Ω is not a countable union of compact subsets, then given any f in $C_0(\Omega)$ one can find a g in $C_0(\Omega)$ such that the support of g is not contained in the support of f. This means that there can be no bounded continuous function b on Ω such that $bf = g$. In other words an algebra $\mathcal{Q}(X)$ of acceptable operators on a B-space X, unlike the algebra of bounded operators on a Hilbert space, need not act transitively on X.

One might still hope, in analogy with Hilbert space *representations of C^*-algebras, to write X as an orthogonal direct sum of subspaces on which $\mathcal{Q}(X)$ does act transitively. However, according to the remarks in §I, if X is such a direct sum of two invariant subspaces, then there are projections in $\mathcal{Q}(X)$ defining these subspaces. But projections in $C_b(\Omega)$ are continuous functions assuming only the values 0 and 1, so that if Ω is connected no trivial projections and hence no such nontrivial decompositions can exist.

III. Induced representations and complete positivity. In this section we will show how a *representation of a pre-C^*-algebra A on a B-space can, under certain circumstances, be used to transfer or "induce" *representations from B to A. Before defining this procedure in general it seems worthwhile to review quickly from this point of view the process of inducing representations of finite groups.

Suppose G is a finite group, H a subgroup of G. Let $C(G)$ and $C(H)$ denote their complex group algebras. If $(\pi, \mathfrak{K})$ is a representation of H (and so of $C(H)$) on a complex vector space $\mathfrak{K}$, then, since $C(H)$ is a subalgebra of $C(G)$, $C(G) \otimes_{C(H)} \mathfrak{K}$ is in a natural way a left $C(G)$-module–the G-module induced by π as defined for instance by Curtis and Reiner in **[5]** or more generally by Higman in **[9]**. A construction analogous to this one was studied by Rieffel in **[13]** for Banach space representations of locally compact groups and Banach algebras. For C^*-algebras the problem, solved by Fell **[8]** and Rieffel **[14]**, is to equip the induced representation space with an inner product in relation to which the induced representation will be a *representation. In order to do this for the case at hand we can use a special case of the device used by Rieffel and Fell. Regarding an element f of $C(G)$ as a complex valued function on G, define $\Phi(f)$ to be the element of $C(H)$ obtained by restricting f to H. Then define a $C(H)$-valued inner product on $C(G)$ by $\langle\langle f_1, f_2\rangle\rangle = \Phi(f_2^* * f_1)$, where $f^*(g) = \bar{f}(g^{-1})$ and $*$ denotes the usual convolution product. Given a positive definite inner product $(\cdot, \cdot)$ on $\mathfrak{K}$ in relation to which π is a unitary representation of H, we can now define a positive semidefinite inner product $\langle\cdot, \cdot\rangle$ on $C(G) \otimes \mathfrak{K}$ by setting

$$\langle f_1 \otimes \chi_1, f_2 \otimes \chi_2\rangle = (\pi(\langle\langle f_1, f_2\rangle\rangle)\chi_1, \chi_2)$$

for f_1 and f_2 in $C(G)$ and χ_1 and χ_2 in $\mathfrak{K}$ and extending. This inner product turns out to be well defined and positive definite on $C(G) \otimes_{C(H)} \mathfrak{K}$ and to have all the desired properties. Now $C(G) \otimes_{C(H)} \mathfrak{K}$ is the quotient of the tensor product by a certain subspace W which is usually defined to be the span of all elements of the form $f \otimes \pi(\gamma)\chi - f * \gamma \otimes \chi$, for all f in $C(G)$, γ in

$C(H)$, χ in $\mathfrak{K}$. From the positive definiteness of the inner product on the quotient we can conclude that W could as well be defined as the subspace of null vectors of the inner product $\langle\cdot, \cdot\rangle$. We will adopt this latter point of view here and presume that the algebraic relationship between $C(G)$ and $C(H)$ is extraneous.

We attempt to perform an analogous construction with arbitrary pre-C^*-algebras A and B replacing $C(G)$ and $C(H)$ and a B-space X, $\langle\langle\cdot, \cdot\rangle\rangle$ with a *representation of A replacing $C(G)$ in the tensor product. Suppose that $(\pi, \mathfrak{K})$ is a *representation of B on a Hilbert space $\mathfrak{K}$. Equip the algebraic tensor product $X \otimes \mathfrak{K}$ with the inner product $\langle\cdot, \cdot\rangle$ determined by

$$\langle x_1 \otimes \chi_1, x_2 \otimes \chi_2\rangle = (\pi(\langle\langle x_1, x_2\rangle\rangle)\chi_1, \chi_2), \qquad x_1, x_2 \text{ in } X, \chi_1, \chi_2 \text{ in } \mathfrak{K}.$$

The quotient of $X \otimes \mathfrak{K}$ by its null vectors will be a pre-Hilbert space if $\langle\cdot, \cdot\rangle$ turns out to be positive semidefinite. However this need not always be the case as the following example shows.

Let A be the C^*-algebra of all 2×2 complex matrices and let $B = A$. Let $\Phi(a) = a^t$, the transpose of a. Let $X = A$ be the definite A-space with inner product $\langle\langle a_1, a_2\rangle\rangle = \Phi(a_2^* a_1) = (a_2^* a_1)^t$. Suppose the *representation $(\pi, \mathfrak{K})$ is just left matrix multiplication on $\mathfrak{K} = \mathbf{C}^2$ with its usual inner product. Then $X \otimes \mathfrak{K}$, under the inner product $\langle\cdot, \cdot\rangle$ defined above, is the direct sum of four orthogonal two-dimensional A-submodules on one of which $\langle\cdot, \cdot\rangle$ is negative definite while being positive definite on the other three. (If e_{ij} denotes the 2×2 matrix with 1 in the ith row, jth column and 0 elsewhere, one can take for the submodules those with the following orthogonal bases: $\{e_{11} \otimes (1, 0), e_{21} \otimes (1, 0)\}$, $\{e_{22} \otimes (0, 1), e_{12} \otimes (0, 1)\}$, $\{e_{11} \otimes (0, 1) + e_{12} \otimes (1, 0), e_{21} \otimes (0, 1) + e_{22} \otimes (1, 0)\}$, and $\{e_{11} \otimes (0, 1) - e_{12} \otimes (1, 0), e_{21} \otimes (0, 1) - e_{22} \otimes (1, 0)\}$, the last being negative definite.) It follows from a proposition below that this complication cannot arise if X, $\langle\langle\cdot, \cdot\rangle\rangle$ is defined by a positive linear map $\Phi\colon A \to B$ which is completely positive with respect to B. Since Størmer [**18**, 6.1], Arveson [**2**, 1.2.2] and Stinespring [**17**, Theorem 4] have shown that Φ must be completely positive if either B or A is commutative, this example is, as noted by Arveson, the simplest possible.

Let X, $\langle\langle\cdot, \cdot\rangle\rangle$ again be an arbitrary B-space but suppose that for every *representation $(\pi, \mathfrak{K})$ of B, the inner product $\langle\cdot, \cdot\rangle$, as defined above, is always positive semidefinite. This means that for any χ in $\mathfrak{K}$, $x_1, \ldots, x_n$ in X and $b_1, \ldots, b_n$ in B,

$$0 \leq \left\langle \sum_{i=1}^n x_i \otimes \pi(b_i)\chi, \sum_{i=1}^n x_i \otimes \pi(b_i)\chi \right\rangle$$
$$= \sum_{i,j} (\pi(\langle\langle x_i, x_j\rangle\rangle)\pi(b_i)\chi, \pi(b_j)\chi) = \left(\pi\left(\sum_{i,j} b_j^*\langle\langle x_i, x_j\rangle\rangle b_i\right)\chi, \chi\right).$$

Since any continuous positive linear functional on B can be written in the form $b \mapsto (\pi(b)\chi, \chi)$ for some $(\pi, \mathfrak{K})$ and χ in $\mathfrak{K}$, we can conclude that $\Sigma_{i,j} b_j^* \langle\langle x_i, x_j \rangle\rangle b_i$ must be a positive element of B for any choice of finite sequences $x_1, \ldots, x_n$ in X and $b_1, \ldots, b_n$ in B. Conversely we can easily see, using the fact that $\pi(B)\mathfrak{K}$ must be dense in $\mathfrak{K}$, that this property implies that the inner products $\langle \cdot, \cdot \rangle$ will always be positive semidefinite. Therefore the following definition and proposition.

Definition. Let B be a pre-C^*-algebra, C a subspace of B, and X, $\langle\langle \cdot, \cdot \rangle\rangle$ a B-space. X will be called completely positive with respect to C if for any finite sequences $x_1, \ldots, x_n$ in X and $b_1, \ldots, b_n$ in C,

$$\sum_{i,j=1}^{n} b_j^* \langle\langle x_i, x_j \rangle\rangle b_i$$

is a positive element of B. If $C = B$ and no confusion should arise X will simply be called completely positive.

When X, $\langle\langle \cdot, \cdot \rangle\rangle$ arises from a positive linear map Φ as in §II, Φ is called completely positive if X is completely positive. In this case the definition of the inner product $\langle \cdot, \cdot \rangle$ and the notion of complete positivity go back to Stinespring **[17]**. (Cf. **[18]**, **[2]** and **[12]**. In **[12]**, Paschke gives a proof that the definition of complete positivity of a linear map relevant here is equivalent to that given in **[17]**.)

Proposition. *Let B be a pre-C^*-algebra and X, $\langle\langle \cdot, \cdot \rangle\rangle$ a B-space. The inner product defined on $X \otimes \mathfrak{K}$ by $\langle x \otimes \chi, y \otimes \xi \rangle = (\pi(\langle\langle x, y \rangle\rangle)\chi, \xi)$ will be positive semidefinite for all *representations $(\pi, \mathfrak{K})$ of B if and only if X, $\langle\langle \cdot, \cdot \rangle\rangle$ is completely positive* (*with respect to B*). □

Theorem. *Let A and B be pre-C^*-algebras and X, $\langle\langle \cdot, \cdot \rangle\rangle$ a completely positive B-space with a *representation of A. If $(\pi, \mathfrak{K})$ is a *representation of B, let $\mathfrak{K}^X$ denote the pre-Hilbert space derived from $X \otimes \mathfrak{K}$ by dividing out the null space of $\langle \cdot, \cdot \rangle$. Then the action of A on $\mathfrak{K}^X$ inherited from $X \otimes \mathfrak{K}$ is well defined and determines a *representation of A* (*which will be denoted $(\pi^X, \mathfrak{K}^X)$*).

Proof. The action of A on $\mathfrak{K}^X$ is well defined since the null vectors of $\langle \cdot, \cdot \rangle$ form an A-submodule–if $x_1, \ldots, x_n$ in X and $\chi_1, \ldots, \chi_n$ in $\mathfrak{K}$ are such that for any y in X and ξ in $\mathfrak{K}$,

$$\left\langle \sum_i x_i \otimes \chi_i, y \otimes \xi \right\rangle = 0$$

then for any a in A,

$$\left\langle \sum_i ax_i \otimes \chi_i, y \otimes \xi \right\rangle = \sum_i (\pi(\langle\langle x_i, a^*y \rangle\rangle)\chi_i, \xi)$$
$$= \left\langle \sum_i x_i \otimes \chi_i, a^*y \otimes \xi \right\rangle = 0.$$

Also, the action of A on $X \otimes \mathfrak{K}$ "preserves adjoints", i.e., if x and y are in X, χ and ξ in $\mathfrak{K}$, then for any a in A

$$\begin{aligned}\langle ax \otimes \chi, y \otimes \xi \rangle &= (\pi(\langle\langle ax, y \rangle\rangle)\chi, \xi) \\ &= (\pi(\langle\langle x, a^*y \rangle\rangle)\chi, \xi) \\ &= \langle x \otimes \chi, a^*y \otimes \xi \rangle.\end{aligned}$$

Adjoints are therefore also preserved on $\mathfrak{K}^X$. In order to conclude that the action of A on $\mathfrak{K}^X$ is a *representation we need to show that the action of each a in A is continuous. First extend $\langle\langle \cdot, \cdot \rangle\rangle$ to the completion $\tilde{X}$ of X, allowing it to take values in the completion of B when necessary, and notice that the extension is completely positive since $\langle\langle \cdot, \cdot \rangle\rangle$ and multiplication in B are continuous. Now extend the *representation of A in X to a *representation of the completion of A on $\tilde{X}$. If η is any element of $X \otimes \mathfrak{K}$, complete positivity implies that $a \mapsto \langle a\eta, \eta \rangle$ is a positive linear functional defined on the completion of A and is therefore bounded with norm $\langle \eta, \eta \rangle$. Consequently $\|a\eta\| \leqslant \|a\| \, \|\eta\|$ so $\eta \mapsto a\eta$ is continuous and $\|\pi^X(a)\| \leqslant \|a\|$. Therefore $(\pi^X, \mathfrak{K}^X)$ is a *representation of A. □

The *representation $(\pi^X, \mathfrak{K}^X)$ will be called the representation of A induced by X and π. Note that in these notations X signifies the vector space X equipped with the particular B-valued inner product and B-space *representation of A under consideration. From the construction it should be apparent that the induced representation depends only on X and the unitary equivalence class of π. Notice also that if X is not definite it makes no difference if X is replaced by its definite quotient space.

The construction of $\mathfrak{K}^X$ above is obviously similar to the construction used by Stinespring in the proof of Theorem 1 of **[17]** but perhaps less obviously similar to that used by Sz.-Nagy in his appendix to **[15]** where tensor products and positive linear maps do not appear explicitly. Nevertheless Sz.-Nagy's théorème principal is an easy corollary of the theorem proved above.

The next proposition shows that we may assume without harm that pre-C^*-algebras and B-spaces are complete. It also gives evidence that the $\|\cdot\|_*$-topology on a B-space is appropriate for dealing with induced representations.

PROPOSITION. *Let A_0 and B_0 be pre-C^*-algebras and let A and B be their completions. Let Y be a completely positive, definite B_0-space with a *-representation of A_0. Let X denote the completion of Y. Then X is a completely* *positive B-space, the *representation of A_0 on Y may be extended to a *representation of A on X, and for any *representation π of B, π^X is unitarily equivalent to π^Y.*

PROOF. We noted earlier that the *representation of A_0 on Y can be extended to a *representation of A on the completion X and that X will be completely positive with respect to B. Now, let $(\pi, \mathfrak{K}_0)$ be a *representation of B_0 on a pre-Hilbert space $\mathfrak{K}_0$ and let $(\pi, \mathfrak{K})$ denote the extension of π to B acting on the completion $\mathfrak{K}$ of $\mathfrak{K}_0$. We may identify $\mathfrak{K}_0^Y$ with a subspace of $\mathfrak{K}^X$ and the actions of A_0 will obviously be intertwined by this identification so it is only necessary to show that $\mathfrak{K}_0^Y$ is dense in $\mathfrak{K}^X$. If it were not, then there would be a sequence of elements η_n of $X \otimes \mathfrak{K}$ such that their images converge in the completion of $\mathfrak{K}^X$ to some element orthogonal to $\mathfrak{K}_0^Y$. In particular, for each y in Y and χ in $\mathfrak{R}_0$ one would have $\langle y \otimes \chi, \eta_n \rangle \to 0$. This will contradict the assumption that Y is dense in X if we can show that the map $x \mapsto \lim\langle x \otimes \chi, \eta_n\rangle$ is a continuous linear functional on X. In order to prove the continuity, first note that for any x in X, χ in $\mathfrak{K}$ and η in $X \otimes \mathfrak{K}$,

$$\begin{aligned} |\langle x \otimes \chi, \eta\rangle| &\leq \langle x \otimes \chi, x \otimes \chi\rangle^{1/2}\langle \eta, \eta\rangle^{1/2} \\ &= (\pi(\langle\langle x, x\rangle\rangle)\chi, \chi)^{1/2}\|\eta\| \\ &\leq \|x\|_*\|\chi\|\ \|\eta\|. \end{aligned}$$

From this we can conclude that the map $x \mapsto \langle x \otimes \chi, \eta\rangle$ is continuous on X and of norm less than or equal to $\|\chi\|\ \|\eta\|$. Since the images of the η_n form a Cauchy sequence in $\mathfrak{K}^X$, this implies that for any fixed χ the sequence of maps $x \mapsto \langle x \otimes \chi, \eta_n\rangle$ is a Cauchy sequence in the dual of X, i.e., $x \to \lim\langle x \otimes \chi, \eta_n\rangle$ defines a continuous linear functional on X. □

It is easy to check that the inducing process respects direct sums, i.e., if X_1 and X_2 are completely positive B-spaces with *representations of A and X is the direct sum of X_1 and X_2 and if the *representation $(\pi, \mathfrak{K})$ of B is the direct sum of $(\pi_1, \mathfrak{K}_1)$ and $(\pi_2, \mathfrak{K}_2)$, then π^X is the direct sum of π^{X_1} and π^{X_2} and also the direct sum of π_1^X and π_2^X.

PROPOSITION. *Let B be a C*-algebra and X a completely positive B-space. Then, regarding B as a subalgebra of W*(B), X is also completely positive with respect to W*(B).*

PROOF. Let $x_1, \ldots, x_n$ be in X, $\beta_1, \ldots, \beta_n$ in $W^*(B)$. We must show that $\Sigma_{i,j}\beta_j^*\langle\langle x_i, x_j\rangle\rangle\beta_i$ is a positive element of $W^*(B)$. Since the positive linear functionals on B separate the points of $W^*(B)$ (see, e.g., **[16**, 1.7.2]) it will suffice to show that if φ is a state of B, then

$$\varphi\left(\sum_{i,j} \beta_j^*\langle\langle x_i, x_j\rangle\rangle\beta_i\right) \geq 0.$$

In fact, since the set of positive linear functionals on B of norm less than or equal to one is the weak* closure of the convex hull of the pure states of B together with 0 (see, e.g., [**6**, 2.5.5]), it will suffice to show that the inequality holds when φ is a pure state. Let $(\pi, \mathfrak{K})$ be the *representation of B determined by φ in the usual way. The fact that φ is a pure state means that this *representation is algebraically irreducible [**6**, 2.5.4, 2.8.4] and so if χ is any nonzero vector of $\mathfrak{K}$, $\pi(B)\chi = \mathfrak{K}$. Now π may be extended to a *representation of $W^*(B)$ on $\mathfrak{K}$ [**16**, 1.21.13] and there is a vector χ_0 in $\mathfrak{K}$ such that $\varphi(\beta) = (\pi(\beta)\chi_0, \chi_0)$. Because X is completely positive with respect to B, we know that for any choice of $b_1, \ldots, b_n$ in B,

$$0 \leqslant \varphi\Big(\sum_{i,j} b_j^* \langle\langle x_i, x_j\rangle\rangle b_i\Big)$$

$$= \sum_{i,j} \big(\pi(\langle\langle x_i, x_j\rangle\rangle)\pi(b_i)\chi_0, \pi(b_j)\chi_0\big)$$

and so, since $\pi(B)\chi_0 = \mathfrak{K}$,

$$0 \leqslant \sum_{i,j} \big(\pi(\langle\langle x_i, x_j\rangle\rangle)\chi_i, \chi_j\big)$$

for any choice of $\chi_1, \ldots, \chi_n$ in $\mathfrak{K}$. In particular, the inequality holds for the choice $\pi(\beta_1)\chi_0, \ldots, \pi(\beta_n)\chi_0$, which produces the desired inequality. □

IV. Induction in stages and the Blattner property. If G is a finite group, H a subgroup of G, K a subgroup of H and π a representation of K, the theorem on induction in stages says that if we induce π from K to H and then induce the resulting representation from H to G we obtain the same representation as by inducing π from K directly to G. For induced representations of C^*-algebras the situation is in general not quite so simple. We will present a version of induction in stages similar to that given by Rieffel in [**14**].

Suppose that A, B and C are C^*-algebras. Let $X, \langle\langle\cdot, \cdot\rangle\rangle$ be a completely positive B-space with a *representation of A and let $Y, \langle\langle\cdot, \cdot\rangle\rangle$ be a completely positive C-space with a *representation of B. If $(\pi, \mathfrak{K})$ is a *representation of C, the *representation $(\pi^Y, \mathfrak{K}^Y)$ induced by π and Y to B may then be induced via X to a *representation $((\pi^Y)^X, (\mathfrak{K}^Y)^X)$ of A. We will consider that we have a theorem on induction in stages if there exists a completely positive C-space Z with a *representation of A such that $(\pi^Z, \mathfrak{K}^Z)$ is unitarily equivalent to $((\pi^Y)^X, (\mathfrak{K}^Y)^X)$.

We begin by attempting to define a C-valued inner product on $X \otimes Y$ by setting

$$((x_1 \otimes y_1, x_2 \otimes y_2)) = \langle\langle(\langle\langle x_1, x_2\rangle\rangle)y_1, y_2\rangle\rangle.$$

This can be obviously be extended in a linear-conjugate linear way to all of $X \otimes Y$, but we must check that it is positive.

LEMMA. *The complete positivity of X implies that $((\cdot, \cdot))$ is positive.*

PROOF. We must check that for any finite sequences $x_1, \ldots, x_n$ in X and $y_1, \ldots, y_n$ in Y, the element $((\Sigma_i x_i \otimes y_i, \Sigma_i x_i \otimes y_i))$ of C is positive. This will follow if the value at this element of any positive linear functional φ on C is nonnegative. Now $\langle y_1, y_2\rangle_\varphi = \varphi(\langle\langle y_1, y_2\rangle\rangle)$ defines a positive semidefinite inner product on Y and the action of B on its definite quotient space defines a *representation of B. Since

$$\varphi\left(\left(\left(\sum_i x_i \otimes y_i, \sum_i x_i \otimes y_i\right)\right)\right) = \sum_{i,j} \varphi(\langle\langle(\langle\langle x_i, x_j\rangle\rangle) y_i, y_j\rangle\rangle)$$
$$= \sum_{i,j} \langle(\langle\langle x_i, x_j\rangle\rangle) y_i, y_j\rangle_\varphi,$$

the positivity follows when we notice that the last term of this equality must be nonnegative if X is completely positive. □

Now define an action of A on $X \otimes Y$ by $a(x \otimes y) = (ax) \otimes y$. It is easy to check that $((az_1, z_2)) = ((z_1, a^*z_2))$ for any a in A and any z_1, z_2 in $X \otimes Y$. In order to conclude that this action defines a *representation of A on the C-space $X \otimes Y$, $((\cdot, \cdot))$ we must also show the following.

LEMMA. $((az, az)) \leqslant \|a\|^2((z, z))$ *for any a in A and z in $X \otimes Y$.*

PROOF. Let φ be any positive linear functional on C. Then for any fixed z in $X \otimes Y$, the map $a \mapsto \varphi(((az, z)))$ is a positive linear functional on A and is therefore bounded and of norm $\varphi(((z, z)))$. Consequently,

$$\varphi(((az, az))) \leqslant \|a\|^2 \varphi(((z, z)))$$

and since φ was arbitrary, the lemma is proved. □

Let Y^X denote the C-space $X \otimes Y$ equipped with the inner product $((\cdot, \cdot))$ and the *representation of A defined above.

THEOREM. *Y^X is completely positive and for any *representation $(\pi, \mathfrak{R})$ of C, $(\pi^{Y^X}, \mathfrak{R}^{Y^X})$ is unitarily equivalent to $((\pi^Y)^X, (\mathfrak{R}^Y)^X)$.*

PROOF. $\mathfrak{R}^Y$ is the quotient of $Y \otimes \mathfrak{R}$ by the null space N of the inner product defined by

$$\langle y_1 \otimes \chi_1, y_2 \otimes \chi_2\rangle = (\pi(\langle\langle y_1, y_2\rangle\rangle)\chi_1, \chi_2),$$

which is positive semidefinite because Y is completely positive. $(\mathfrak{R}^Y)^X$ is the quotient of $X \otimes ((Y \otimes \mathfrak{R})/N)$ by the null space M of the inner product defined by

$$[x_1 \otimes (y_1 \otimes \chi_1 + N), x_2 \otimes (y_2 \otimes \chi_2 + N)]$$
$$= \langle \pi(\langle\langle x_1, x_2\rangle\rangle)(y_1 \otimes \chi_1 + N), (y_2 \otimes \chi_2 + N)\rangle$$
$$= (\pi(\langle\langle(\langle\langle x_1, x_2\rangle\rangle) y_1, y_2\rangle\rangle)\chi_1, \chi_2),$$

which is positive semidefinite because X is completely positive. If we now define a map U from $(X \otimes Y) \otimes \mathfrak{K}$ onto $(\mathfrak{K}^Y)^X$ by

$$U(x \otimes y \otimes \chi) = x \otimes (y \otimes \chi + N) + M,$$

we see that the inner product defined on $Y^X \otimes \mathfrak{K}$ by $\langle z_1, z_2 \rangle = [U(z_1), U(z_2)]$ is identical to the usual inner product. Consequently, the latter must also be positive semidefinite. Since this must be true for any *representation $(\pi, \mathfrak{K})$ of C, we are entitled to conclude that Y^X must be completely positive. Finally the map defined by U from the definite quotient space $\mathfrak{K}^{Y^X}$ onto $(\mathfrak{K}^Y)^X$ preserves inner products and intertwines the two induced representations of A. □

This proposition allows us to make a connection with the pre-Hilbert B-modules of Paschke **[12]** and the right B-rigged spaces of Rieffel **[14]**. These differ only slightly from the objects satisfying the following definition.

DEFINITION. Let B be a pre-C^*-algebra. A B-space X, $\langle\langle \cdot, \cdot \rangle\rangle$ will be called a B-*module if B acts on X from the right in such a way that X is a right B-module in the usual algebraic sense and if in addition $\langle\langle x, yb \rangle\rangle = b^*\langle\langle x, y \rangle\rangle$ and $\langle\langle xb, y \rangle\rangle = \langle\langle x, y \rangle\rangle b$ for all x and y in X and b in B.

Notice that a B-*module is always completely positive since for any $x_1, \ldots, x_n$ in X and $b_1, \ldots, b_n$ in B,

$$\sum_{i,j} b_j^* \langle\langle x_i, x_j \rangle\rangle b_i = \sum_{i,j} \langle\langle x_i b_i, x_j b_j \rangle\rangle = \left\langle\!\!\left\langle \sum_i x_i b_i, \sum_i x_i b_i \right\rangle\!\!\right\rangle.$$

With this property in mind one might attempt to generalize the preceding definition by requiring merely that for any given x and b there exist at least one element $x \cdot b$ in X such that $\langle\langle x, y \rangle\rangle b = \langle\langle x \cdot b, y \rangle\rangle$ and (consequently) $b^*\langle\langle y, x \rangle\rangle = \langle\langle y, x \cdot b \rangle\rangle$ for all y in X. However, by choosing Hamel bases for X and B and for each x and b in the bases setting $xb = x \cdot b$ for some such $x \cdot b$, we can obtain by extending linearly a bilinear map $X \times B \to X$ possessing all the properties required to determine a B-*module except perhaps that $x(b_1 b_2) = (xb_1)b_2$. If X, $\langle\langle \cdot, \cdot \rangle\rangle$ is definite

$$\begin{aligned}\langle\langle xb_1b_2 - (xb_1)b_2, y \rangle\rangle &= \langle\langle x, y \rangle\rangle b_1 b_2 - \langle\langle xb_1, y \rangle\rangle b_2 \\ &= \langle\langle x, y \rangle\rangle b_1 b_2 - \langle\langle x, y \rangle\rangle b_1 b_2 \\ &= 0\end{aligned}$$

for any y in X, from which we can conclude that $xb_1b_2 = (xb_1)b_2$. Since the *representations induced by a B-space and those induced by its definite quotient are identical, this generalization would gain little.

Notice that any pre-C^*-algebra may be considered to be a B-*module if we equip it with the inner product $\langle\langle b_1, b_2 \rangle\rangle = b_2^* b_1$. If X is any completely positive B-space, we can then make B^X into a B-*module by setting $(x \otimes$

$b)b_1 = x \otimes (bb_1)$. The preceding theorem shows that X and B^X will induce the same *representations.

Besides being always completely positive, the B-*modules possess some other nice properties. Among them is the Blattner property defined below. In addition the acceptable operators on a definite B-*module X are B-linear, i.e., if T is an acceptable operator on X, then $T(xb) = T(x)b$ for any x in X and b in B. This is a consequence of the following manipulations:

$$\langle\langle T(xb), y\rangle\rangle = \langle\langle xb, T^*(y)\rangle\rangle = \langle\langle x, T^*(y)\rangle\rangle b$$
$$= \langle\langle T(x), y\rangle\rangle b = \langle\langle T(x)b, y\rangle\rangle.$$

DEFINITION. Let B be a pre-C^*-algebra and X a completely positive B-space. If φ is a bounded positive linear functional on B, let $(\pi_\varphi, \mathfrak{R}_\varphi)$ be the *representation of B determined by φ. If A is a pre-C^*-algebra with a *representation on X, let $(\pi_\varphi^X, \mathfrak{R}_\varphi^X)$ denote the *representation of A induced from π_φ via X. We may obtain another *representation of A by defining the inner product $\langle x_1, x_2\rangle_\varphi = \varphi(\langle\langle x_1, x_2\rangle\rangle)$ on X. The action of A on the definite quotient space X_φ then determines a *representation of A. X will be said to have the Blattner property if for any pre-C^*-algebra A with a *representation on X and for any bounded positive linear functional φ on B the *representation of A on X_φ is unitarily equivalent to $(\pi_\varphi^X, \mathfrak{R}_\varphi^X)$.

The Blattner property as defined above is inspired by the analogous property which holds for induced representations of locally compact groups as shown by Blattner in [3]. It does not hold for all induced representations as defined here. When it does hold we can give a version of induction in stages somewhat more directly analogous to the situation for groups.

COROLLARY. *Let A, B and C be C^*-algebras. Let X be a completely positive B-space with *representation of A and possessing the Blattner property. Let Φ: $B \to C$ be a bounded positive linear map such that the C-space constructed from Φ and B, as in* §II, *is completely positive and has the Blattner property. If $(\pi, \mathfrak{R})$ is the *representation of C determined by the positive linear functional φ on C, then the twice induced *representation $((\pi^B)^X, (\mathfrak{R}^B)^X)$ of A is unitarily equivalent to the *representation of A on $X_{\varphi\circ\Phi}$.*

PROOF. Because the C-space B, with inner product determined by Φ, has the Blattner property, the induced *representation $(\pi^B, \mathfrak{R}^B)$ is unitarily equivalent to the *representation of B determined by the positive linear functional $\varphi \circ \Phi$. Because X has the Blattner property, the *representation $((\pi^B)^X, (\mathfrak{R}^B)^X)$ of A induced by the latter *representation of B must be unitarily equivalent to the *representation of A on $X_{\varphi\circ\Phi}$. □

Although the proof and the terminology are somewhat different, the following proposition is essentially the same as Rieffel's Corollary 5.5 in [14].

It establishes that the construction of induced representations used here is, when applied to *modules, identical in result to the construction used by Rieffel.

PROPOSITION. *Let B be a pre-C^*-algebra, X a B-*module. Then X has the Blattner property.*

PROOF. If B does not have an identity, we may adjoin one and make X a *module of the larger algebra by setting $x1 = x$. Since we have a one-one correspondence between the *representations of B and the *representations of B with an identity adjoined, we may as well assume that B has an identity. Let φ be any positive linear functional on B and let $(\pi, \mathfrak{K})$ be the *representation of B determined by φ. Choose an element χ of $\mathfrak{K}$ such that $\varphi(b) = (\pi(b)\chi, \chi)$ for all b in B. Then $\pi(B)\chi$ is a dense B-submodule of $\mathfrak{K}$ and, as noted earlier, we may therefore replace $\mathfrak{K}$ by $\pi(B)\chi$ in the construction of $\mathfrak{K}^X$. Now define the map U from $X \otimes \pi(B)\chi$ onto X by $U(x \otimes \pi(b)\chi) = xb$. Inner products are preserved by U since

$$\begin{aligned}\langle x_1 \otimes \pi(b_1)\chi, x_2 \otimes \pi(b_2)\chi\rangle &= (\pi(\langle\langle x_1, x_2\rangle\rangle b_1)\chi, \pi(b_2)\chi)\\ &= \varphi(b_2^*\langle\langle x_1, x_2\rangle\rangle b_1)\\ &= \varphi(\langle\langle x_1b_1, x_2b_2\rangle\rangle).\end{aligned}$$

U will therefore determine a well-defined unitary map on the definite quotient spaces. If A is any pre-C^*-algebra with a *representation on X, it is apparent that U will intertwine the actions of A on $X \otimes \pi(B)\chi$ and X so that we can conclude that the *representation $(\pi^X, \mathfrak{K}^X)$ of A is unitarily equivalent to the *representation of A on X_φ. □

The proposition below shows that in the finite dimensional case the Blattner property and the *module property are even more intimately related than the preceding proposition indicates. The general case seems to be more complicated.

PROPOSITION. *Let B be a finite dimensional C^*-algebra and X a finite dimensional, completely positive, definite B-space. Then X has the Blattner property if and only if X is a B-*module.*

PROOF. Suppose that X has the Blattner property. Since B is finite dimensional we can define the trace $\mathrm{Tr}(b)$ of any element b of B. The inner product defined on B by the positive linear functional Tr is positive definite and, since X, $\langle\langle \cdot, \cdot \rangle\rangle$ is definite, the inner product $\mathrm{Tr}(\langle\langle \cdot, \cdot \rangle\rangle)$ is also positive definite. The Blattner property implies that there is a linear map U of $X \otimes B$ onto X which preserves inner products, i.e., such that for any x and y in X and b and c in B,

$$\mathrm{Tr}(\langle\langle U(x \otimes b), U(y \otimes c)\rangle\rangle) = \mathrm{Tr}(c^*\langle\langle x, y\rangle\rangle b).$$

Given an x in X and b in B, we want to show that there must be an element z in X such that $\langle\langle z, y\rangle\rangle = \langle\langle x, y\rangle\rangle b$ for any y in X, or, equivalently, such that

$$\mathrm{Tr}(c^*\langle\langle z, y\rangle\rangle) = \mathrm{Tr}(c^*\langle\langle x, y\rangle\rangle b)$$

for any c in B and y in X. This equality may be rewritten as

$$\mathrm{Tr}(\langle\langle U(z \otimes 1), U(y \otimes c)\rangle\rangle) = \mathrm{Tr}(\langle\langle U(x \otimes b), U(y \otimes c)\rangle\rangle),$$

so that in order to conclude that such a z exists we need only show that $U(X \otimes 1) = X$. But U is faithful on the subspace $X \otimes 1$ since $U(x \otimes 1) = 0$ implies that

$$\mathrm{Tr}(\langle\langle x, x\rangle\rangle) = \mathrm{Tr}(\langle\langle U(x \otimes 1), U(x \otimes 1)\rangle\rangle) = 0,$$

so the existence of z follows from the finite dimensionality of X. From the remarks following the definition we know that this is enough to define a B-*module action of B on X.

Since we have already seen that being a B-*module implies that X has the Blattner property, the proof is complete. □

V. Weak containment. Of the several characterizations of weak containment, to be found for instance in **[6]** and **[7]**, the following will be most convenient here. Let A be a C^*-algebra. A *representation π of A is said to be weakly contained in a set S of *representations of A if the kernel of π contains the intersection of the kernels of the *representations of S, i.e., $\ker \pi \supset \bigcap_{\sigma \in S} \ker \sigma$. Let $\hat{A}$ denote the set of all unitary equivalence classes of irreducible *representations of A. If S is a set of *representations of A, we will mean by the support of S the set of all equivalence classes $[\pi]$ in $\hat{A}$ such that π is weakly contained in S. If S is a subset of $\hat{A}$, the support of the *representations associated with elements of S is called the weak closure of S. The weakly closed sets determine the usual topology on $\hat{A}$ (cf. **[6**, 3.4.11**]**).

Let B be another and X, $\langle\langle \cdot, \cdot\rangle\rangle$ a completely positive B-space with a *representation of A. The following theorem on the "continuity" of the inducing process is a generalization of Fell's Theorem 4.2 in **[7]** for the usual representations of separable locally compact groups (see also the special case treated in Rieffel's Proposition 6.26 of **[14]**).

THEOREM. *Let S be a set of *representations of B. If a *representation $(\pi, \mathfrak{H})$ of B is weakly contained in S, then $(\pi^X, \mathfrak{H}^X)$ is weakly contained in the set $S^X = \{\sigma^X\colon \sigma \in S\}$ of all *representations of A induced via X from the members of S.*

PROOF. Suppose that an element a of A is in the kernel of σ^X for each σ in S. We want to show that a is necessarily in the kernel of π^X. For any σ in S our assumption is that

$$(\sigma(\langle\langle ax, ax\rangle\rangle)\xi_1, \xi_2) = \langle ax \otimes \xi_1, ax \otimes \xi_2\rangle = 0$$

for any x in X and any ξ_1 and ξ_2 in the Hilbert space on which σ acts, i.e., $\sigma(\langle\langle ax, ax\rangle\rangle) = 0$ for all x in X. Since this is true for all σ in S and since π is weakly contained in S we must have $\pi(\langle\langle ax, ax\rangle\rangle) = 0$ for all x in X. Therefore

$$\langle ax \otimes \chi, ax \otimes \chi\rangle = (\pi(\langle\langle ax, ax\rangle\rangle)\chi, \chi) = 0$$

for any x in X and any χ in $\mathfrak{K}$, which implies that a is in the kernel of π^X. □

If φ is a positive linear functional on B, X_φ will denote as before the definite quotient space obtained from X equipped with the inner product $\langle\cdot, \cdot\rangle_\varphi = \varphi(\langle\langle\cdot, \cdot\rangle\rangle)$. The action of A on each X_φ determines a *representation and we will use $\hat{X}$ to denote the support of the set of all such *representations obtained as φ runs through the states of B. Let ker X be the set of all elements a of A such that $\langle\langle ax, ax\rangle\rangle = 0$ for all x in X (so if $\langle\langle\cdot, \cdot\rangle\rangle$ is definite, ker X is the set of all a such that $ax = 0$ for all x in X). Note that ker X equals the intersection of the kernels of the *representations of A on the X_φ as φ runs through the states of B. Consequently, the unitary equivalence class of an irreducible *representation σ of A lies in $\hat{X}$ if and only if ker $\sigma \supset$ ker X. Finally, let $\hat{B}^X$ denote the set of all *representations of A induced via X from irreducible *representations of B. Since every *representation of B is weakly contained in the set of all irreducible *representations, the theorem shows that the support of $\hat{B}^X$ contains the supports of all *representations of A weakly contained in any *representation induced from B via X.

THEOREM. *The support of $\hat{B}^X$ is $\hat{X}$.*

PROOF. Let π be a *representation of B. If an element a of A is in ker X, then

$$\langle ax \otimes \chi, ax \otimes \chi\rangle = (\pi(\langle\langle ax, ax\rangle\rangle)\chi, \chi) = 0$$

for any x in X and any χ in the Hilbert space on which π acts. Consequently, a must be in ker π^X. Thus ker $\pi^X \supset$ ker X so π^X must be weakly contained in $\hat{X}$. Since $\hat{X}$ is weakly closed, we can conclude from the preceding theorem that the support of $\hat{B}^X$ is contained in $\hat{X}$.

Conversely, if a is in ker π^X for all irreducible *representations π of B, then for any x in X and for each such π

$$(\pi(\langle\langle ax, ax\rangle\rangle)\chi, \xi) = \langle ax \otimes \chi, ax \otimes \xi\rangle = 0$$

for all χ and ξ in the Hilbert space of π, so $\pi(\langle\langle ax, ax\rangle\rangle) = 0$ for each such π and therefore $\langle\langle ax, ax\rangle\rangle = 0$. Thus ker X contains the intersection over all irreducible π of the kernels ker π^X, from which we can conclude that $\hat{X}$ is contained in the support of $\hat{B}^X$. □

COROLLARY. *The support of $\hat{B}^X$ is all of $\hat{A}$ if and only if* $\ker X = \{0\}$.

PROOF. From the proof of the preceding theorem we know that $\ker X = \bigcap \ker \pi^X$, the intersection being taken over all irreducible *representations π of B. If $\ker X = \{0\}$, then this intersection is $\{0\}$ and so is contained in the kernel of any irreducible *representation of A, which implies that $\hat{A}$ is contained in the support of $\hat{B}^X$. Conversely, if $\hat{A}$ is contained in the support of $\hat{B}^X$ this implies that the above intersection is $\{0\}$ and consequently so is $\ker X$. □

Note that we are assuming in this section that A is a C^*-algebra. The assumption that A is complete cannot be dropped.

VI. Completely positive maps of finite dimensional C^*-algebras. Throughout this section A and B will denote two finite dimensional C^*-algebras. A completely positive linear map Φ from A into B determines a completely positive B-space with a *representation of A as in §§II and III. If π is a *representation of B, let $\Phi(\pi)$ denote the *representation of A induced by π via this B-space. If Φ' is another such map we will say that Φ and Φ' are inductively equivalent if the *representations $\Phi(\pi)$ and $\Phi'(\pi)$ are unitarily equivalent for all *representations π of B. We will find a complete collection of representatives of the inductive equivalence classes.

We first reduce the problem to the case of simple algebras. Let $E_1, \ldots, E_p$ be the minimal central projections of A, $F_1, \ldots, F_q$ those of B. If Φ is a linear map from A into B, let Φ_i^j denote the map from A into BF_j given by $\Phi_i^j(a) = \Phi(aE_i)F_j$. It is easy to see that if Φ is completely positive then each Φ_i^j will be completely positive with respect to B. For each $j = 1, \ldots, q$ let σ_j be an irreducible *representation of BF_j. We will identify σ_j with the corresponding *representation of B so that any nondegenerate *representation of B will be unitarily equivalent to a direct sum of multiples of the σ_j.

LEMMA. *Let Φ be a completely positive linear map of A into B and let π be a *representation of B. If π is unitarily equivalent to the direct sum $\bigoplus_j n_j\sigma_j$ then $\Phi(\pi)$ is unitarily equivalent to $\bigoplus_{ij} n_j\Phi_i^j(\sigma_j)$.*

PROOF. Since the inducing process respects direct sums of *representations, it will suffice to show that, for any $j = 1, \ldots, q$, $\Phi(\sigma_j)$ is equivalent to $\bigoplus_i \Phi_i^j(\sigma_j)$. One can easily check that multiplication by any E_i defines an acceptable operator on the B-space determined by A and Φ and, since the E_i are selfadjoint idempotents whose sum is the identity, this B-space is the direct sum of the B-spaces determined by A and the completely positive maps $\Phi_i = \sum_j \Phi_i^j$. Since the inducing process respects direct sums of B-spaces, $\Phi(\sigma_j)$ must be equivalent to the direct sum $\bigoplus_i \Phi_i(\sigma_j)$. But, for each i and j, $\sigma_j \circ \Phi_i = \sigma_j \circ \Phi_i^j$ and from this we can conclude that $\Phi_i(\sigma_j)$ is identical to $\Phi_i^j(\sigma_j)$. Therefore $\Phi(\sigma_j)$ is equivalent to $\bigoplus_i \Phi_i^j(\sigma_j)$. □

Since the E_i are mutually orthogonal, $\Phi_i^j(\sigma_j)$ is zero on the ideals AE_k for which $k \neq i$. On the basis of this and the lemma we can restrict attention to simple algebras.

Let M and N be the C^*-algebras of $m \times m$ and $n \times n$ matrices with complex entries. We will identify the tensor product algebra $M \otimes N$ with the C^*-algebra of $mn \times mn$ matrices. Also we will identify $M \otimes N$ with the vector space of linear maps from M into N in the following manner. Let e_{ij} be the $m \times m$ matrix with 1 in the ith row, jth column and 0 elsewhere. Let f_{xy} be the similar $n \times n$ matrix. If Φ is a linear map of M into N, let the corresponding element of $M \otimes N$ be $\Sigma_{i,j} e_{ij} \otimes \Phi(e_{ij})$. If $\Sigma_{ijxy} h_{ij}^{xy} e_{ij} \otimes f_{xy}$ is any element of $M \otimes N$, then the value of the corresponding map at e_{st} is $\Sigma_{xy} h_{ij}^{xy} f_{xy}$. Also, let (ρ, V) be a faithful *representation of M on the m-dimensional Hilbert space V, $(\cdot, \cdot)$ and let $e_1, \ldots, e_m$ be an orthonormal basis of V for which $\rho(e_{ij})e_s = \delta_{js} e_i$ where δ_{js} is 1 if $j = s$ and 0 otherwise. Let (σ, W) and $f_1, \ldots, f_n$ be similar objects for N.

LEMMA. *A linear map Φ from M into N is completely positive if and only if the corresponding element of $M \otimes N$ is positive.*

PROOF. Let $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ be elements of V and W and suppose that, for each p, $v_p = \Sigma_i v_p^i e_i$. Then

$$\begin{aligned}
&\left(\rho \otimes \sigma\left(\sum_{ij} e_{ij} \otimes \Phi(e_{ij})\right) \sum_p v_p \otimes w_p, \sum_q v_q \otimes w_q\right) \\
&\qquad = \sum_{ijpq} \big(\rho(e_{ij})v_p, v_q\big)\big(\sigma(\Phi(e_{ij}))w_p, w_q\big) \\
&\qquad = \sum_{ijpq} v_p^j \bar{v}_p^i \big(\sigma(\Phi(e_{ij}))w_p, w_q\big) \\
&\qquad = \sum_{pq} \left[\sigma\left(\Phi\left(\sum_{ij} v_p^j \bar{v}_q^i e_{ij}\right)\right) w_p, w_q\right] \\
&\qquad = \sum_{pq} \left[\sigma\left(\Phi\left(\left(\sum_i v_q^i e_{si}\right)^* \left(\sum_i v_p^i e_{si}\right)\right)\right) w_p, w_q\right]
\end{aligned}$$

for any choice of $s = 1, \ldots, m$. If Φ is completely positive, the last line must be nonnegative. Since $v_1, \ldots, v_k$ and $w_1, \ldots, w_k$ are arbitrary finite sequences, this implies that $\rho \otimes \sigma(\Sigma_{ij} e_{ij} \otimes \Phi(e_{ij}))$ is a positive operator on $V \otimes W$ and since $\rho \otimes \sigma$ is a faithful *representation we can conclude that $\Sigma_{ij} e_{ij} \otimes \Phi(e_{ij})$ is a positive element of $M \otimes N$. Conversely, if this element is positive then the last line must always be nonnegative and since any element of M may be written as a sum of elements of the form $\Sigma_i v^i e_{si}$ we can conclude that Φ is completely positive. □

LEMMA. *Let Φ be the completely positive linear map from M into N corresponding to the positive element η of $M \otimes N$. Let u be a unitary in $M \otimes N$ and let Φ^u be the completely positive map corresponding to $u^*\eta u$. Then Φ and Φ^u are inductively equivalent.*

PROOF. It will of course be enough to establish the equivalence of $\Phi(\sigma)$ and $\Phi^u(\sigma)$. The *representation $\Phi(\sigma)$ acts on the definite quotient space, $\Phi(W)$, of $M \otimes W$ equipped with the inner product

$$\langle a_1 \otimes w_1, a_2 \otimes w_2 \rangle = \big(\sigma(\Phi(a_2^* a_1))w_1, w_2\big).$$

Now $M \otimes W$ is, under the action of M, the direct sum of mn copies of a cyclic representation algebraically equivalent to (ρ, V), so the quotient $\Phi(W)$ must be the direct sum of a certain number of copies of (ρ, V), the number being determined by the dimension of $\Phi(W)$. Consequently, in order to establish the unitary equivalence of $\Phi(\sigma)$ and $\Phi^u(\sigma)$ we need only show that the dimensions of $\Phi(W)$ and $\Phi^u(W)$ are equal. Suppose that Φ corresponds to the element $\eta = \sum_{ijxy} h_{ij}^{xy} e_{ij} \otimes f_{xy}$ of $M \otimes N$ and consider the values of the inner product $\langle \cdot, \cdot \rangle$ determined by Φ on basis elements of $M \otimes W$.

$$\begin{aligned}\langle e_{ij} \otimes f_k, e_{pq} \otimes f_r \rangle &= \big(\sigma(\Phi(e_{qp} e_{ij}))f_k, f_r\big) = \big(\sigma(\Phi(\delta_{pi} e_{qj}))f_k, f_r\big) \\ &= \delta_{pi}\left(\sigma\left(\sum_{xy} h_{qj}^{xy} f_{xy}\right) f_k, f_r\right) \\ &= \delta_{pi} \sum_{xy} h_{qj}^{xy} \delta_{xr} \delta_{yk} = \delta_{pi} h_{qj}^{rk}.\end{aligned}$$

Notice first that if $p \neq i$ the inner product is 0 and that $\langle e_{ij} \otimes f_k, e_{iq} \otimes f_r \rangle$ does not depend on i. This means that $M \otimes W$ is the orthogonal direct sum of m copies of an mn-dimensional semidefinite inner product space, say the one spanned by $e_{1j} \otimes f_k, j = 1, \ldots, m;\ k = 1, \ldots, n$. From the form of the values of the inner product on these basis elements it is clear that the dimension of the definite quotient space is rank η, the number of positive eigenvalues of η counted according to multiplicity. The dimension of $\Phi(W)$ is therefore m rank η and $\Phi(\sigma)$ is equivalent to rank η copies of ρ. Since rank $\eta = \mathrm{rank}(u^*\eta u)$ for any unitary u, $\Phi(\sigma)$ is equivalent to $\Phi^u(\sigma)$. □

For any completely positive linear map Φ from M into N let rank $\Phi =$ rank η, where η is the corresponding element of $M \otimes N$. We can always find a unitary u in $M \otimes N$ so that $u^*\eta u$ is diagonal, say $\eta = \sum_{ij} \alpha_i^j e_{ii} \otimes f_{jj}$. If we let φ_i denote the positive linear functional on M given by $\varphi_i(a) = \mathrm{Tr}(e_{ii} a)$, then

$$\Phi^u = \sum_{ij} \alpha_i^j \varphi_i f_{jj} = \sum_j \left(\sum_i \alpha_i^j \varphi_i\right) f_{ii}$$

and rank Φ = rank $\Phi^u = \Sigma_{ij}\mathrm{sign}(\alpha_i^j)$. From the lemma we know that any completely positive map from M into N must be inductively equivalent to one of the maps of this form.

Returning to the general situation, the following is a consequence of the preceding lemmas and remarks.

THEOREM. *Let A and B be finite dimensional C^*-algebras as before. Let σ_j be a (representative of the) nontrivial equivalence class of irreducible *representations of B corresponding to the minimal ideal BF_j and let ρ_i be a similar object for A corresponding to AE_i. Let Φ be a completely positive map of A into B. Then $\Phi(\sigma_j)$ is unitarily equivalent to $\bigoplus_i$ (rank $\Phi_i^j)\rho_i$ where rank Φ_i^j is the rank of Φ_i^j as a map of the simple algebra AE_i into the simple algebra BF_j. Consequently, if π is a *representation of B equivalent to $\bigoplus_j n_j\sigma_j$ then $\Phi(\pi)$ is equivalent to $\bigoplus_j \bigoplus_i n_i(\text{rank } \Phi_i^j)\rho_i = \bigoplus_i (\Sigma_j n_j \text{ rank } \Phi_i^j)\rho_i$.* □

If we define $\Phi^*(\rho_i)$, for each ρ_i, to be the *representation $\bigoplus_j(\text{rank } \Phi_i^j)\sigma_j$ and set $\Phi^*(\bigoplus_i n_i\rho_i) = \bigoplus_i n_i\Phi^*(\rho_i)$, then we have the following reciprocity theorem.

COROLLARY. *For any nondegenerate *representations μ of A and π of B,*

$$\mathrm{Hom}_A(\Phi(\pi), \mu) = \mathrm{Hom}_B(\pi, \Phi^*(\mu)).$$

PROOF. It will suffice to verify this for $\pi = \sigma_j$ and $\mu = \rho_i$.

$$\begin{aligned}\mathrm{Hom}_A(\Phi(\sigma_j), \rho_i) &= \mathrm{Hom}_A\Big(\bigoplus_k (\text{rank } \Phi_k^j)\rho_k, \rho_i\Big)\\ &= (\text{rank } \Phi_i^j)\mathrm{Hom}_A(\rho_i, \rho_i)\\ &= (\text{rank } \Phi_i^j)\mathrm{Hom}_B(\sigma_j, \sigma_j)\\ &= \mathrm{Hom}_B\Big(\sigma_j, \bigoplus_r (\text{rank } \Phi_i^r)\sigma_r\Big)\\ &= \mathrm{Hom}_B(\sigma_j, \Phi^*(\rho_i)).\ \square\end{aligned}$$

Consider now the case in which A and B are the complex group algebras $C(G)$ and $C(H)$ of a finite group G and its subgroup H. Let Φ be the positive linear map defined at the beginning of §III, which gives the usual induced representations of finite groups. If we equip $C(G)$ with the inner product

$$\begin{aligned}(f_1, f_2) &= \frac{1}{|G|} \sum_{g\in G} f_1(g)\overline{f_2(g)}\\ &= \sum_i (\dim \rho_i)\mathrm{Tr}(\rho_i(f_2^*)\rho_i(f_1)),\end{aligned}$$

then Φ is just the orthogonal projection of $C(G)$ onto $C(H)$. If σ_j is an irreducible *representation of $C(H)$ corresponding to the minimal ideal BF_j, then $\Phi(\sigma_j)$ is equivalent to $\bigoplus_i$ (rank $\Phi_i^j)\rho_i$, the summand which is nontrivial on a minimal ideal AE_i being (rank $\Phi_i^j)\rho_i$. Choosing matrices to represent elements of $C(G)$ appropriately, arrange that the matrices representing elements of BF_j are sums of a certain number of identical square matrices (say $p \times p$). Let r_i^j be the number of such blocks which fall in AE_i. The positive linear functional on $C(H)$ given by $\varphi(b) = b_{11}$, where b_{11} is the upper left entry in this matrix representation, gives a *representation equivalent to σ_j. Since the B-space in question has the Blattner property, as shown in [**14**] and in the next section, $\Phi(\sigma_j)$ is equivalent to the *representation of A determined by the positive linear functional $p \circ \Phi$. Since Φ is the orthogonal projection and since the inner product on AE_i is a multiple of the one given by the trace, the value of $\varphi \circ \Phi$ at an element a of AE_i will be a multiple of a sum of r_i^j diagonal entries of the matrix representing a. The *representation of A determined by this positive linear functional will therefore be equivalent to $r_i^j\rho_i$. From this we can conclude that $r_i^j = \text{rank } \Phi_i^j$. Furthermore, it is clear that the nondegenerate part of the restriction of ρ_i to BF_j is equivalent to $r_i^j\sigma_j$ and since the restriction of ρ_i to $C(H)$ is the direct sum of these over all j, we have shown that

$$\rho_i|_{C(H)} = \bigoplus_j r_i^j\sigma_j = \Phi^*(\rho_i).$$

COROLLARY (FROBENIUS). *For the nondegenerate *representations* μ *of* $C(G)$ *and* π *of* $C(H)$,

$$\text{Hom}_{C(G)}(\Phi(\pi), \mu) = \text{Hom}_{C(B)}(\pi, \mu|_{C(H)}).$$

VII. Two examples for groups. The first example will treat the usual Mackey-Blattner induced representations for locally compact groups in essentially the same manner as in Rieffel [**14**] and Fell [**8**]. In the second example we will take a somewhat different approach based on the assumption that the subgroup is compact.

Let G be any locally compact (Hausdorff topological) group. Denote by d_G a left invariant Haar measure on G and by Δ_G its modular function. Let $\mathcal{K}(G)$ be the space of compactly supported, continuous, complex-valued functions on G. We will confuse an element f of $\mathcal{K}(G)$ with the corresponding bounded measure fd_G. $\mathcal{K}(G)$ then forms a *algebra under convolution and the involution given by $f^*(g) = \Delta_G(g^{-1})\bar{f}(g^{-1})$. This *algebra is dense in $C^*(G)$, the C^*-algebra of G (cf. [**6**, 13.9]), and so is a pre-C^*-algebra. Let H be a closed subgroup of G, make definitions analogous to the above, and let D be the function on H given by $D(h) = \Delta_G(h)^{1/2}\Delta_H(h)^{-1/2}$.

Define the linear map Φ from $\mathcal{K}(G)$ into $\mathcal{K}(H)$ by $\Phi(f)(h) = D(h)f(h)$. In order to conclude that Φ is positive we must show, for any f in $\mathcal{K}(G)$, that $\Phi(f^* * f)$ is a positive element of the C^*-algebra of H, i.e., that for any continuous positive definite function φ on H

$$\int \Phi(f^{*}*f)\varphi\, d_H = \int (f^{*}*f)D\varphi\, d_H$$

is nonnegative. This follows from Blattner's theorem in [3] which states, among other things, that if φ is a continuous positive definite function on H, then $D\varphi d_H$ defines a positive definite measure on G. (Blattner uses some nonstandard definitions in [3]. Proofs using terminology identical to that used here can be found in [14, 4.4] and [20, 5.1.2.1].)

The $\mathcal{K}(H)$-space determined by Φ is just $\mathcal{K}(G)$ equipped with the inner product $\langle\langle f_1, f_2\rangle\rangle = Df_2^{*}*f_1|_H$. Although Φ need not be bounded, it must always be relatively bounded. This follows from Blattner's theorem in the following way. Let f be in $\mathcal{K}(G)$. Then we want to show that there is a number k such that for any f_1 in $\mathcal{K}(G)$ we have the inequality

$$\begin{aligned}\langle\langle f_1*f, f_1*f\rangle\rangle &= Df^{*}*f_1^{*}*f_1*f|_H \\ &\leqslant kDf^{*}*f|_H = k\langle\langle f, f\rangle\rangle.\end{aligned}$$

So, for any continuous positive definite function φ on H, we want

$$\int \langle\langle f_1*f, f_1*f\rangle\rangle\varphi\, d_H \leqslant k\int \langle\langle f, f\rangle\rangle\varphi\, d_H.$$

But $\int\langle\langle f, f\rangle\rangle\varphi\, d_H$ is just the inner product on $\mathcal{K}(G)$ determined by the positive definite measure $D\varphi d_H$ so that left convolution determines a *representation of $\mathcal{K}(G)$ on this positive semidefinite inner product space, i.e.,

$$\int \langle\langle f_1*f, f_1*f\rangle\rangle\varphi\, d_H \leqslant \left(\|f_1\|^2_{C^*(G)}\right)\int \langle\langle f, f\rangle\rangle\varphi\, d_H$$

where $\|f_1\|_{C^*(G)}$ denotes the norm of f_1 as an element of $C^*(G)$. So Φ is relatively bounded and we therefore have a *representation of $\mathcal{K}(G)$ on the $\mathcal{K}(H)$-space $\mathcal{K}(G), \langle\langle\cdot, \cdot\rangle\rangle$.

We can make $\mathcal{K}(G)$ into a $\mathcal{K}(H)$-*module by defining, for any f in $\mathcal{K}(G)$ and θ in $\mathcal{K}(H)$,

$$(f\theta)(g) = f*(D^{-1}\theta)(g) = \int f(gh)\theta(h^{-1})D(h)\, d_H h.$$

This action $f, \theta \mapsto f\theta$ is clearly bilinear. It makes $\mathcal{K}(G)$ a right $\mathcal{K}(H)$-module since

$$\begin{aligned}f(\theta_1*\theta_2)(g) &= \int f(gh)\theta_1*\theta_2(h^{-1})D(h)\, d_H h \\ &= \int\int f(gh)\theta_1(h^{-1}k)\theta_2(k^{-1})D(h)\, d_H k\, d_H h \\ &= \int\int f(gkh)\theta_1(h^{-1})\theta_2(k^{-1})D(kh)\, d_H h\, d_H k \\ &= \int (f\theta_1)(gk)\theta_2(k^{-1})D(k)\, d_H k \\ &= ((f\theta_1)\theta_2)(g).\end{aligned}$$

This is a *module action since

$$\begin{aligned}\langle\langle f\theta, f_1\rangle\rangle(h) &= \int D(h)f_1^*(hg)(f\theta)(g^{-1})\,d_G g \\ &= \iint D(h)f_1^*(hg)f(g^{-1}k)\theta(k^{-1})D(k)\,d_H k\,d_G g \\ &= \iint D(hk)f_1^*(hkg)f(g^{-1})\theta(k^{-1})\,d_G g\,d_H k \\ &= \int \langle\langle f, f_1\rangle\rangle(hk)\theta(k^{-1})\,d_H k \\ &= \langle\langle f, f_1\rangle\rangle * \theta(h).\end{aligned}$$

Since $\mathcal{K}(G)$, $\langle\langle\cdot,\cdot\rangle\rangle$ is a $\mathcal{K}(H)$-*module it must have the Blattner property so that if φ is a continuous positive definite function on H and π is the *representation of $\mathcal{K}(H)$ derived from the left regular representation when $\mathcal{K}(H)$ is equipped with the inner product $(\theta_1, \theta_2) = \int \theta_2^* * \theta_1 \varphi\, d_H$, then the *representation π^G of $\mathcal{K}(G)$ induced from π via the $\mathcal{K}(H)$-space $\mathcal{K}(G)$ is unitarily equivalent to the *representation obtained from the left regular representation of $\mathcal{K}(G)$ when $\mathcal{K}(G)$ is equipped with the inner product

$$\langle f_1, f_2\rangle = \int \langle\langle f_1, f_2\rangle\rangle \varphi\, d_H = \int f_2^* * f_1 D\varphi\, d_H.$$

According to another part of the theorem of Blattner mentioned above, the *representation so obtained from the positive definite measure $D\varphi d_H$ is unitarily equivalent to the usual induced representation.

For the second example replace H by the compact subgroup K and let d_K be a Haar measure of total mass 1. Let π be an irreducible continuous unitary representation of K on some Hilbert space, let χ be its character, and let $\eta = (\dim \pi)\check{\chi} d_K$, where $\check{\chi}(k) = \chi(k^{-1})$. Then η is a projection in the measure algebra of G and has compact support. For any f in $\mathcal{K}(G)$, $\eta * f * \eta$ is (identifiable with) an element of $\mathcal{K}(G)$. Define $\Phi(f) = \eta * f * \eta$ and let $\mathcal{K}(G, \pi)$ denote the image of $\mathcal{K}(G)$ under Φ. Then $\mathcal{K}(G, \pi)$ is a *-subalgebra of $\mathcal{K}(G)$, hence a pre-C^*-algebra when equipped with the $C^*(G)$ norm, and Φ is a bounded positive linear map of $\mathcal{K}(G)$ onto $\mathcal{K}(G, \pi)$.

Let $\langle\langle\cdot,\cdot\rangle\rangle$ be the $\mathcal{K}(G, \pi)$-valued inner product on $\mathcal{K}(G)$ given by $\langle\langle f_1, f_2\rangle\rangle = \Phi(f_2^* * f_1)$. Since Φ is bounded we obtain a $\mathcal{K}(G, \pi)$-space with a *representation of $\mathcal{K}(G)$. Defining the right action $f\theta = f * \theta$, for f in $\mathcal{K}(G)$ and θ in $\mathcal{K}(G, \pi)$, makes $\mathcal{K}(G)$ a $\mathcal{K}(G, \pi)$-*module since, for any f_1 in $\mathcal{K}(G)$,

$$\langle\langle f * \theta, f_1\rangle\rangle = \eta * f_1^* * f * \theta * \eta = \eta * f_1^* * f * \eta * \theta = \langle\langle f, f_1\rangle\rangle * \theta$$

(making use of the fact that η is idempotent and $\eta * \theta * \eta = \theta$).

We now define a *representation $(\rho, \mathfrak{R})$ of $\mathcal{K}(G, \pi)$ such that the *representation ρ^G of $\mathcal{K}(G)$ induced from ρ via $\mathcal{K}(G)$, $\langle\langle\cdot,\cdot\rangle\rangle$ is unitarily equivalent to the *representation π^G induced from π via $\mathcal{K}(G)$ in the sense of the preceding example. Choose a continuous positive definite function φ on K associated with π and such that φ is equal to 1 at the identity of K. Equip $\mathcal{K}(G, \pi)$ with the inner product $\langle \theta_1, \theta_2 \rangle = \int \theta_2^* * \theta_1 \varphi \, d_K$ and let $\mathfrak{K}$ denote the pre-Hilbert space obtained by dividing out the null vectors of $\langle \cdot, \cdot \rangle$ (from Blattner's theorem we know that $\langle \cdot, \cdot \rangle$ is positive semidefinite so this makes sense). For any θ in $\mathcal{K}(G, \pi)$ we let $\rho(\theta)$ act on $\mathfrak{K}$ by convolution, that is $\rho(\theta)$ takes the equivalence class in $\mathfrak{K}$ of any θ' in $\mathcal{K}(G, \pi)$ to the equivalence class of $\theta * \theta'$. It is easy to check that ρ defines a *representation of $\mathcal{K}(G, \pi)$, in fact it is a subrepresentation of the restriction of π^G to $\mathcal{K}(G, \pi)$.

Since $\mathcal{K}(G)$ is a $\mathcal{K}(G, \pi)$-*module and $(\rho, \mathfrak{K})$ is the *representation determined by the bounded positive linear functional $\theta \mapsto \int \theta \varphi \, d_K$ the induced *representation ρ^G will be unitarily equivalent to the *representation of $\mathcal{K}(G)$ determined by the positive linear functional $f \mapsto \int \Phi(f) \varphi \, d_K$. In order to conclude that this *representation is unitarily equivalent to π^G we need only to show that

$$\int \Phi(f) \varphi \, d_K = \int f \varphi \, d_K$$

for any f in $\mathcal{K}(G)$. Making use of the facts that $\chi(k^{-1}k_1) = \chi(k_1 k^{-1})$ for any k and k_1 in K and $\chi * \varphi = (\dim \pi)^{-1} \varphi$, we have

$$\begin{aligned}
\int \Phi(f) \varphi \, d_K &= \int \eta * f * \eta(k) \varphi(k) \, d_K k \\
&= \iiint (\dim \pi)^2 \check{\chi}(k k_1 k_2) f(k_2^{-1}) \check{\chi}(k_1^{-1}) \varphi(k) \, d_K k_2 \, d_K k_1 \, d_K k \\
&= \iiint (\dim \pi)^2 \chi(k_2^{-1} k_1^{-1} k^{-1}) f(k_2^{-1}) \chi(k_1) \varphi(k) \, d_K k_2 \, d_K k_1 \, d_K k \\
&= \iiint (\dim \pi)^2 \chi(k^{-1} k_2^{-1} k_1^{-1}) \chi(k_1) f(k_2^{-1}) \varphi(k) \, d_K k_1 \, d_K k_2 \, d_K k \\
&= \iint (\dim \pi) \chi(k^{-1} k_2^{-1}) \varphi(k) f(k_2^{-1}) \, d_K k_2 \, d_K k \\
&= \iint (\dim \pi) \chi(k_2^{-1} k^{-1}) \varphi(k) f(k_2^{-1}) \, d_K k \, d_K k_2 \\
&= \int \varphi(k_2^{-1}) f(k_2^{-1}) \, d_K k_2 \\
&= \int f \varphi \, d_K.
\end{aligned}$$

We can also determine the *representations of $\mathcal{K}(G)$ induced from other *representations of $\mathcal{K}(G, \pi)$. Any cyclic *representation of $\mathcal{K}(G, \pi)$ must be constructed by a bounded positive linear functional ψ defined on the closure in $C^*(G)$ of $\mathcal{K}(G, \pi)$. Since Φ is continuous we can extend it in a unique way to a continuous positive linear map defined on all $C^*(G)$ and taking its values in the closure of $\mathcal{K}(G, \pi)$. Then $\psi \circ \Phi$ is a continuous positive linear functional on $C^*(G)$ so there must be a continuous positive definite function φ on G such that $\psi \circ \Phi(f) = \int f\varphi \, d_G$ for all f in $\mathcal{K}(G)$. Now for any f in $\mathcal{K}(G)$, we have $\Phi(f) = \Phi(\Phi(f))$ so that

$$\int f\varphi \, d_G = \int \Phi(f)\varphi \, d_G = \int \eta * f * \eta\varphi \, d_G$$

$$= \int\int\int (\dim \pi)^2 \check{\chi}(k_1) f(g) \check{\chi}(k_2) \varphi(k_1 g k_2) \, d_K k_1 \, d_K k_2 \, d_G g$$

$$= \int (\dim \pi)^2 \chi * \varphi * \chi(g) f(g) \, d_G g$$

$$= \int \check{\eta} * \varphi * \check{\eta} f \, d_G.$$

From the fact that this is true for any f in $\mathcal{K}(G)$ we can conclude that $\check{\eta} * \varphi * \check{\eta} = \varphi$. So every cyclic *representation of $\mathcal{K}(G, \pi)$ can be obtained from a positive linear functional of the form $\theta \mapsto \int \theta\varphi \, d_G$ where φ is a continuous positive definite function on G bi-invariant under convolution by $\check{\eta}$. From the fact that $\mathcal{K}(G)$, $\langle\langle \cdot, \cdot \rangle\rangle$ possesses the Blattner proper, we can easily conclude that if σ is the *representation of $\mathcal{K}(G, \pi)$ determined by such a φ, then σ^G is unitarily equivalent to the *representation of $\mathcal{K}(G)$ determined by φ in the usual manner.

We can, of course, make all of the above definitions and conclusions for any of the irreducible continuous unitary representations of K. We differentiate among them by using a subscript π with those associated with a particular representation π, e.g., η_π and Φ_π. Also for each π, let X_π denote the completion of the definite $\mathcal{K}(G, \pi)$-space determined by $C^*(G)$ and Φ_π. Then, in the terminology of §V, the weak closure of the union of the subsets $\hat{X}_\pi$ of $C^*(G)\hat{}$ as π runs through the equivalence classes of irreducible unitary representations is all of $C^*(G)\hat{}$. In order to prove this we only need to show that the only element of $C^*(G)$ which lies in ker X_π for all π is 0. We can then invoke the second theorem of §V to obtain the desired conclusion. Suppose an element a of $C^*(G)$ is in each ker X_π. Then for any b in $C^*(G)$ and any π, $\Phi_\pi((ab)^*(ab)) = \langle\langle ab, ab \rangle\rangle_\pi = 0$. This implies that $ab = 0$ as can be seen from the following argument. Let $(\sigma, \mathfrak{H})$ be a faithful, nondegenerate *representation of $C^*(G)$. Since $(\sigma, \mathfrak{H})$ also determines a nondegenerate representation of K, $\mathfrak{H}$ is the orthogonal direct sum, over all equivalence classes of irreducible representations π of K_1, of its subspaces $\sigma(\eta_\pi)\mathfrak{H}$. Now if $\Phi_\pi((ab)^*(ab)) = 0$, then

$$\left(\sigma(\eta_\pi(ab)^*(ab)\eta_\pi)\mathfrak{H}, \mathfrak{H}\right) = \{0\}$$

so

$$(\sigma(ab)\sigma(\eta_\pi)\mathfrak{H}, \sigma(ab)\sigma(\eta_\pi)\mathfrak{H}) = \{0\}.$$

If this is true for all π, then $\sigma(ab) = 0$ and consequently $ab = 0$. Since $ab = 0$ for all b in $C^*(G)$ and since $C^*(G)$ contains an approximate identity, we have $a = 0$. Thus the intersection of all the ideals ker X_π is $\{0\}$ and therefore the kernel of any irreducible *representation of $C^*(G)$ contains this intersection. Consequently, the weak closure of the supports $\hat{X}_\pi$ is all of $C^*(G)^\wedge$.

VIII. The bounded positive linear maps on group algebras. Let G be a locally compact group with left Haar measure d_G and let $L^1(G) = L^1(G, d_G)$ be its group algebra (consisting of equivalence classes of functions). $L^1(G)$ may be identified with a dense *subalgebra of $C^*(G)$ and since the $C^*(G)$-norm of an element of $L^1(G)$ is no larger than its $L^1(G)$-norm, the bounded linear maps defined on $L^1(G)$, which we deal with here, include the bounded linear maps on $C^*(G)$.

Any positive linear functional on $L^1(G)$ may be written in the form $f \mapsto \int f\theta \, d_G$ where θ is a continuous positive definite function on G. In proving this one can make use of the duality between $L^1(G)$ and $L^\infty(G)$, as for example in [**6**, 13.4.5]. We are concerned here with giving a similar representation for bounded positive linear maps of $L^1(G)$ into a C^*-algebra. Although the convenient duality of the complex valued case is not in general available, the proposition established below for positive maps will be enough for our purpose.

We will use the following notion of vector valued integral. Let B be a Banach space, B^* its Banach space dual, and μ a (complex or extended-real valued) measure. A function F from G into B will be called μ-integrable if for each τ in B^* the complex valued function $\tau \circ F$ is μ-integrable and if $\tau \mapsto \int \tau \circ F \, d\mu$ defines a (possibly unbounded) linear functional on B^*. This linear functional will be denoted $\int F \, d\mu$.

If F is a function from G into a Banach space B, F will be called bounded if its norm $\|F\|$ is finite, where $\|F\| = \sup_{g\in G}\|F(g)\|$, $\|F(g)\|$ being the norm in B of $F(g)$. If B is the dual of a unique Banach space B_*, we will say that F is weak* continuous if it is a continuous map of G into B when B is equipped with its weak* $(\sigma(B, B_*))$ topology.

It will be frequently convenient to identify, without comment, an element of a Banach space with its canonical image in the bidual.

PROPOSITION. *Let B be a W^*-algebra and let Φ be a bounded positive linear map of $L^1(G)$ into B. Then there is a bounded weak* continuous function θ from G into B such that $\theta(f) = \int f\theta \, d_G$ for any f in $L^1(G)$ and such that $\|\theta\| = \|\Phi\|$.*

PROOF. Let B_* be the predual of B. The equality $\Phi(f) = \int f\theta \, d_G$ is intended to mean that for any τ in B_*, $\tau \circ \Phi(f) = \int \tau \circ \theta f \, d_G$. If τ is positive, $\tau \circ \Phi$ is a bounded positive linear functional on $L^1(G)$ so there is a continuous positive definite function θ_τ on G such that $\tau \circ \Phi(f) = \int f\theta_\tau \, d_G$. Also $\|\tau \circ \Phi\| = \theta_\tau(e) = \|\theta_\tau\|$, where e is the identity of G and $\|\theta_\tau\|$ is the supremum norm of θ_τ. Now if τ is any element of B_*, we can write it as a finite linear combination $\tau = \sum \lambda_i \tau_i$ of positive elements and define $\theta_\tau = \sum \lambda_i \theta_{\tau_i}$. Then $\tau \circ \Phi(f) = \int f\theta_\tau \, d_G$. θ_τ is uniquely defined since it is continuous and its class in $L^\infty(G)$ is unique. Also $\|\tau \circ \Phi\| = \|\theta_\tau\|$. If g is an element of G, then

$$|\theta_\tau(g)| \leqslant \|\tau \circ \Phi\| \leqslant \|\tau\| \, \|\Phi\|$$

so the map $\tau \mapsto \theta_\tau(g)$ is a continuous linear functional on B_*. Consequently, there is a unique element $\theta(g)$ in B such that $\theta_\tau(g) = \tau \circ \theta(g)$ $(= \theta(g)(\tau))$. The weak* continuity of $g \mapsto \theta(g)$ follows from the fact that $g \mapsto \theta_\tau(g)$ is continuous for any τ in B_*. Finally, for any g in G,

$$\|\theta(g)\| = \sup_{\tau \in B_*} \frac{|\theta_\tau(g)|}{\|\tau\|} \leqslant \sup_{\tau \in B_*} \frac{\|\tau \circ \Phi\|}{\|\tau\|}$$

$$\leqslant \|\Phi\| = \sup_{f \in L^1(G)} \frac{\|\Phi(f)\|}{\|f\|} = \sup_{f \in L^1(G)} \frac{\|\int f\theta \, d_G\|}{\|f\|} \leqslant \|\theta\|,$$

from which we can conclude that $\|\theta\| = \|\Phi\|$. □

Notice that if B is only a C^*-algebra, we can apply the proposition to $W^*(B)$.

Let $M(G)$ be the Banach *algebra of bounded measures on G.

Corollary. *If B is a W^*-algebra and Φ is a bounded positive linear map of $L^1(G)$ into B, then Φ may be extended to a positive linear map $\tilde{\Phi}$ of equal norm from $M(G)$ into B.*

Proof. Let μ be in $M(G)$. We can define the element $\tilde{\Phi}(\mu)$ in B by $\tau \circ \tilde{\Phi}(\mu) = \int \theta_\tau \, d\mu$, for all τ in B_*, since

$$|\tau \circ \tilde{\Phi}(\mu)| = \left| \int \theta_\tau \, d\mu \right| \leqslant \|\theta_\tau\| \, \|\mu\|$$

$$\leqslant \|\tau\| \, \|\theta\| \, \|\mu\| = \|\tau\| \, \|\Phi\| \, \|\mu\|.$$

Clearly $\tilde{\Phi}$ is linear and $\|\tilde{\Phi}\| = \|\Phi\|$. If μ is a positive element of $M(G)$ and τ is a positive element of B_*, then $\tau \circ \tilde{\Phi}(\mu) = \int \theta_\tau \, d_\mu \geqslant 0$ since θ_τ is positive definite on G. Consequently $\tilde{\Phi}$ is positive. □

The proposition and analogy with the complex valued case suggest the following definition.

Definition. Let B be a C^*-algebra. A function θ from G into B will be called positive definite if $\sum_i \lambda_i \bar{\lambda}_j \theta(g_i g_j^{-1})$ is a positive element of B for any finite sequences $g_1, \ldots, g_n$ of elements of G and $\lambda_1, \ldots, \lambda_n$ of complex numbers.

If θ is positive definite and τ in B^* is positive, then $\tau \circ \theta$ is a positive definite complex valued function so $|\tau \circ \theta(g)| \leqslant \tau \circ \theta(e)$ for any g in G. Since any element τ of B^* can be written in the form $\Sigma_{i=1}^4 \lambda_i \tau_i$, where for each i, τ_i is positive, $\|\tau_i\| \leqslant \|\tau\|$, and $|\lambda_i| \leqslant 1$, we can easily conclude that $\|\theta(g)\| \leqslant 4\|\theta(e)\|$ for any g in G. Thus every positive definite function is bounded.

The following definition seems to be appropriate when dealing with completely positive linear maps.

DEFINITION. Let B be a C^*-algebra, C a linear subspace of B. A function θ from G into B will be called completely positive with respect to C if $\Sigma_{ij} b_j^* \theta(g_i g_j^{-1}) b_i$ is a positive element of B for any finite sequences $g_1, \ldots, g_n$ of elements of G and $b_1, \ldots, b_n$ of elements of C.

Notice that the above definition is just a generalization of the notions of positive definiteness for operator valued functions as used by Kunze **[10]** and Sz.-Nagy **[19]**.

From an argument similar to that employed in the proof of the final proposition of §III, it follows that if θ: $G \to W^*(B)$ is completely positive with respect to B, then θ is completely positive with respect to $W^*(B)$. Since $W^*(B)$ always has an identity we can easily see that complete positivity with respect to B implies positive definiteness.

If B is a C^*-algebra with identity 1 and θ: $G \to B$ is completely positive with respect to the subspace spanned by its range and 1, then $\|\theta\| = \|\theta(e)\|$. The proof is essentially the same as for the complex valued case. The complete positivity with respect to 1 insures that θ is positive definite and this implies $\theta(g^{-1}) = \theta(g)^*$. The complete positivity also implies that

$$\theta(g)^*\theta(gg^{-1})\theta(g) + \theta(g)^*\theta(g)\bar{\lambda}1 + \lambda\theta(g^{-1})\theta(g) + \lambda\theta(e)\bar{\lambda}1$$

must be positive for any g in G and any complex number λ. So if τ in B^* is positive and λ is real,

$$\tau(\theta(g)^*\theta(e)\theta(g)) + 2\tau(\theta(g)^*\theta(g))\lambda + \tau(\theta(e))\lambda^2 \geqslant 0.$$

Since $\theta(e)$ is positive, $\tau(\theta(e)) \leqslant \|\tau\| \; \|\theta(e)\|$ and

$$\tau(\theta(g)^*\theta(e)\theta(g)) \leqslant \tau(\theta(g)^*\theta(g))\|\theta(e)\|,$$

so to insure the positivity of the quadratic in λ we must have

$$\tau(\theta(g)^*\theta(g))^2 \leqslant \tau(\theta(g)^*\theta(e)\theta(g))\tau(\theta(e))$$
$$\leqslant \|\tau\|\tau(\theta(g)^*\theta(g))\|\theta(e)\|^2.$$

Consequently

$$\|\theta(g)\|^2 = \sup_{\substack{\tau \geqslant 0 \\ \|\tau\| = 1}} \tau(\theta(g)^*\theta(g)) \leqslant \|\theta(e)\|^2,$$

so that $\|\theta\| = \|\theta(e)\|$.

PROPOSITION. *Let B be a W^*-algebra and Φ a bounded linear map of $L^1(G)$ into B. Then Φ is completely positive if and only if $\Phi(f) = \int f\theta \, d_G$ for all f in $L^1(G)$, where θ is a weak* continuous map of G into B which is completely positive with respect to B.*

PROOF. Suppose Φ is completely positive. Then Φ is also positive and so can be represented in the required form with θ weak* continuous and positive definite. Given $g_1, \ldots, g_n$ in G and $b_1, \ldots, b_n$ in B, we must show that for any positive τ in B_*,

$$\tau\left(\sum b_j^* \theta\left(g_i g_j^{-1}\right) b_i\right) \geqslant 0.$$

Let $\{U_\alpha\}$ be a collection of compact neighborhoods of e in G directed by containment and with intersection $\{e\}$. For each $i = 1, \ldots, n$ let χ_i^α be the characteristic function of the set $g_i U_\alpha$. Then for any complex valued continuous function f on G,

$$\lim_\alpha \int f \chi_i^\alpha * (\chi_j^\alpha)^* \, d_G = f\left(g_i g_j^{-1}\right).$$

So for any positive τ in B_*,

$$\begin{aligned}\tau\left(\sum_{ij} b_j^* \theta\left(g_i g_j^{-1}\right) b_i\right) &= \sum_{ij} \tau\left(b_j^* \theta\left(g_i g_j^{-1}\right) b_i\right) \\ &= \lim_\alpha \sum_{ij} \int \tau(b_j^* \theta b_i) \chi_i^\alpha * (\chi_j^\alpha)^* \, d_G \\ &= \lim_\alpha \sum_{ij} \tau\left(b_j^* \left(\int \theta \chi_i^\alpha * (\chi_j^\alpha)^* \, d_G\right) b_i\right) \\ &= \lim_\alpha \sum_{ij} \tau\left(b_j^* \Phi\left(\chi_i^\alpha * (\chi_j^\alpha)^*\right) b_i\right) \\ &\geqslant 0.\end{aligned}$$

Conversely, suppose that $\theta\colon G \to B$ is weak* continuous and completely positive with respect to B. Then θ defines a bounded linear map Φ of $M(G)$ into B via $\Phi(\mu) = \int \theta \, d\mu$. We must show that for any $\mu_1, \ldots, \mu_n$ in $M(G)$ and $b_1, \ldots, b_n$ in B, $\sum_{ij} b_j^* \Phi(\mu_i * \mu_j^*) b_i$ is positive. Let τ in B_* be positive. Then

$$\begin{aligned}\tau\left(\sum_{ij} b_j^* \Phi(\mu_i * \mu_j^*) b_i\right) &= \sum_{ij} \tau\left(b_j^* \left(\int \theta \, d(\mu_i * \mu_j^*)\right) b_i\right) \\ &= \int \left(\sum_{ij} \tau(b_j^* \theta b_i)\right) d(\mu_i * \mu_j^*).\end{aligned}$$

If each μ_i has finite support, say $\mu_i = \sum_k \lambda_{ik} \varepsilon_{ik}$ where ε_{ik} is the unit point mass at an element g_{ik}, then

$$\int\left(\sum_{ij}\tau(b_j^*\theta b_i)\right)d(\mu_i*\mu_j^*)=\sum_{ij}\tau\left(b_j^*\left(\sum_{ks}\lambda_{ik}\bar{\lambda}_{js}\sigma(g_{ik}g_{js}^{-1})\right)b_i\right)$$
$$=\tau\left(\sum_{ijks}(\lambda_{js}b_j)^*\theta(g_{ik}g_{js}^{-1})(\lambda_{ik}b_i)\right)$$

which must be nonnegative because θ is completely positive. Since $g\mapsto\sum_{ij}\tau(b_j^*\theta(g)b_i)$ is a bounded continuous function,

$$\int\left(\sum_{ij}\tau(b_j^*\theta b_i)\right)d(\mu_i*\mu_j^*)$$

is, for any $\mu_1,\ldots,\mu_n$ in $M(G)$, a limit of such finite expressions, hence is nonnegative. □

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